On Maxwell's Conjecture for Coulomb Potential Generated by Point Charges

Alexei Yu. Uteshev[(✉)] and Marina V. Yashina

Faculty of Applied Mathematics, St. Petersburg State University,
Universitetskij pr. 35, Petrodvorets, 198504 St. Petersburg, Russia
alexeiuteshev@gmail.com, marina.yashina@gmail.com

Abstract. The problem discussed herein is the one of finding the set of stationary points for the Coulomb potential function $F(P) = \sum_{j=1}^{K} m_j/|PP_j|$ for the cases of $K = 3$ and $K = 4$ positive charges $\{m_j\}_{j=1}^{K}$ fixed at the positions $\{P_j\}_{j=1}^{K} \subset \mathbb{R}^2$. Our approach is based on reducing the problem to that of evaluation of the number of real solution of an appropriate algebraic system of equations. We also investigate the bifurcation picture in the parameter domains.

Keywords: Coulomb potential · Stationary points · Maxwell's conjecture

1 Introduction

Given the coordinates of $K \geq 3$ points $\{P_j\}_{j=1}^{K} \subset \mathbb{R}^3$, find the coordinates of stationary points for the function

$$F(P) = \sum_{j=1}^{K} \frac{m_j}{|PP_j|} \ . \tag{1}$$

Here $\{m_j\}_{j=1}^{K}$ are assumed to be real non-zero numbers and $|\cdot|$ stands for the Euclidian distance.

This problem can be viewed as a classical electrostatics one with the function (1) representing the Coulomb potential of the charges $\{m_j\}_{j=1}^{K}$ placed at fixed (stationary) positions $\{P_j\}_{j=1}^{K}$ in the space. Thus, the stated problem can be considered as an origin for the general problem of simulation or motions of charged particles in electric or magnetic fields; the stationary point of the potential corresponds then to the equilibrium position of a probe particle. On the other hand, the function (1) can also be interpreted as the Newton (gravitational) potential with $\{m_j\}_{j=1}^{K}$ treated as masses fixed at $\{P_j\}_{j=1}^{K}$. Despite its classical looking formulation, the problem has not been given a systematic exploration — with the exception of some special configurations like the one treated in [3] where the points $\{P_j\}_{j=1}^{K}$ make an equilateral polygon and all the charges $\{m_j\}_{j=1}^{K}$ are assumed to be equal. The difficulty of the problem can be acknowledged also from the state of the art with its part known as

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Maxwell's Conjecture [7]**.** The total number of stationary points of any configuration with K charges in $\mathbb{R}^3$ never exceeds $(K-1)^2$.

This conjecture was investigated in [4,8] with the aid of some topological principles. However, it remains still open even for the case of $K=3$ equal charges.

The coordinates of stationary points of the function (1) satisfy the system of equations

$$\frac{DF}{DP} = \mathbb{O} \qquad \Longleftrightarrow \qquad \sum_{j=1}^{K} \frac{m_j(P-P_j)}{|PP_j|^3} = \mathbb{O}\,. \tag{2}$$

Solving this system with the aid of numerical iteration methods, like the gradient descent one, can lead one to poor convergence due to the unbounded growth of iteration values when the solution being searched lies in a close neighborhood of a charge location.

The present paper is devoted to an alternative approach for solving the system (2) based on symbolic computations. We first intend *to eliminate radicals* from the system (2), i.e. to reduce it to a system polynomially dependent on the coordinates of the point P. This can be done in different ways, and therefore it is quite reasonable to look for the procedure which can diminish the degrees of the final algebraic equations. For this purpose, we are going to exploit an approach suggested in the paper [11] where the general problem of finding the stationary point set for the function $F(P) = \sum_{j=1}^{K} m_j \left|PP_j\right|^L$ for arbitrary values of the exponent $L \ne 0$ was treated. Then, for the obtained algebraic system, we solve the problem of *localization of* its *solutions*, i.e. we aim at finding the true number of real solutions and separating them. The mathematical background for this approach is based on the technique of *elimination of variables* from the algebraic systems with the aid of *the resultant* computation. Our analysis of the behavior of the stationary point set in its dependency on the parameters involved into the problem (like the values of charges or coordinates of charge location) will be essentially based on the *discriminant* sign evaluation. We refer the reader to [2,5,10] for brushing up some basic results of Elimination Theory utilized in the foregoing sections.

We will deal here only with the cases of potential generated by 3 or 4 positive charges located on the plane. One possible misunderstanding should be cleared out in connection with this assumption. In some examples given below, we use the expressions *minimum* or *stable* stationary point. The *Earnshaw's theorem* states that a collection of point charges in $\mathbb{R}^3$ cannot be maintained in a stable stationary equilibrium configuration solely by the electrostatic interaction of the charges [9]. Therefore, the term *stability* hereinafter should be understood as the *conditional stability* in the plane of charges location.

2 Three Points

Let the points $\{P_j = (x_j, y_j)\}_{j=1}^{3}$ be noncollinear, i.e. the determinant

$$S = \left| \begin{matrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{matrix} \right| \tag{3}$$

does not vanish. For definiteness, we will assume hereinafter that the points P_1, P_2, P_3 are counted counterclockwise, i.e. the determinant (3) is positive.

Stationary points of the Coulomb potential

$$F(P) = \frac{m_1}{|PP_1|} + \frac{m_2}{|PP_2|} + \frac{m_3}{|PP_3|} \tag{4}$$

are given by the system of Eq. (2) which, for this particular case, can be written down as

$$\left\{ \begin{array}{l} \dfrac{m_1(x-x_1)}{|PP_1|^3} + \dfrac{m_2(x-x_2)}{|PP_2|^3} + \dfrac{m_3(x-x_3)}{|PP_3|^3} = 0, \\ \dfrac{m_1(y-y_1)}{|PP_1|^3} + \dfrac{m_2(y-y_2)}{|PP_2|^3} + \dfrac{m_3(y-y_3)}{|PP_3|^3} = 0. \end{array} \right. \tag{5}$$

In order to transform this system into an algebraic one, the straightforward approach can be utilized consisting in successive squaring of the equations and eliminating the radicals one by one. If one denotes by A_1, A_2 and A_3 the summands in any of the above equations, then this procedure is executed as follows

$$A_1+A_2+A_3=0 \quad \Rightarrow \quad (A_1+A_2)^2 = A_3^2 \quad \Rightarrow \quad (2\,A_1A_2)^2 = (A_3^2 - A_1^2 - A_2^2)^2\,.$$

However this approach (tackled in [6] for finding a boundary for the number of stationary points) results in a drastic increase of the order and complexity of the final algebraic equations. The resulting system can be reduced to an algebraic one

$$F_1(x,y,m_1,m_2,m_3)=0,\ F_2(x,y,m_1,m_2,m_3)=0$$

where F_1, F_2 are polynomials of the degree 28 with respect to the variables x and y (and with the coefficients of the orders up to 10^{19} for Example 1 treated below). Finding all the real solutions of this system with the aid of elimination of variable procedure (resultant or the Gröbner basis computation [2]) is a hardly feasible task.

An alternative approach for reducing the system (5) to an algebraic one was suggested in [11]. It is based on the following result:

Theorem 1. *Set*

$$S_1(x,y) = \left| \begin{array}{ccc} 1 & 1 & 1 \\ x & x_2 & x_3 \\ y & y_2 & y_3 \end{array} \right|,\ S_2(x,y) = \left| \begin{array}{ccc} 1 & 1 & 1 \\ x_1 & x & x_3 \\ y_1 & y & y_3 \end{array} \right|,\ S_3(x,y) = \left| \begin{array}{ccc} 1 & 1 & 1 \\ x_1 & x_2 & x \\ y_1 & y_2 & y \end{array} \right|\,. \tag{6}$$

Any solution of the system (5) is a solution of the system

$$m_1 : m_2 : m_3 = |PP_1|^3 S_1(x,y) : |PP_2|^3 S_2(x,y) : |PP_3|^3 S_3(x,y)\,. \tag{7}$$

The underlying idea of the proof of this theorem is simple: system (5) can be treated as a linear one with respect to the parameters m_1, m_2, m_3 and therefore can be resolved with the aid of Cramer's formulae, which are equivalent to (7).

Squaring the ratio (7) gives rise to the following algebraic system

$$m_2^2 S_1^2 |PP_1|^6 - m_1^2 S_2^2 |PP_2|^6 = 0,\ m_2^2 S_3^2 |PP_3|^6 - m_3^2 S_2^2 |PP_2|^6 = 0\,. \tag{8}$$

Example 1. Let $P_1 = (1,1), P_2 = (5,1), P_3 = (2,6)$. Analyse the structure of the set of stationary points of the potential (4) for $m_1 = 1$ and for m_2, m_3 treated as parameters.

Solution. The system (8) is as follows

$$\begin{cases} \tilde{F}_1(x,y,m_2,m_3) = (5\,x+3\,y-28)^2(x^2+y^2-2\,x-2\,y+2)^3m_2^2 \\ \qquad\qquad\qquad\quad -(5\,x-y-4)^2(x^2+y^2-10\,x-2\,y+26)^3 = 0, \\ \tilde{F}_2(x,y,m_2,m_3) = (4\,y-4)^2(x^2+y^2-4\,x-12\,y+40)^3m_2^2 \\ \qquad\qquad\qquad\quad -m_3^2(5\,x-y-4)^2(x^2+y^2-10\,x-2\,y+26)^3 = 0 \end{cases} \tag{9}$$

and the degree of $\tilde{F}_1$ and $\tilde{F}_2$ with respect to the variables x and y equals 8. This time, in comparison with the approach mentioned at the beginning of the present section, it is realistic to eliminate any variable from this system. For instance, the resultant of these polynomials treated with respect to x

$$\mathcal{Y}(y,m_2,m_3) = \mathcal{R}_x(\tilde{F}_1,\tilde{F}_2) \tag{10}$$

is[1] the polynomial of the degree 34 in y . For any specialization of parameters m_2 and m_3, it is possible to find the exact number of real zeros and to localize the latter in the ideology of symbolic computations, e.g., via the Sturm series construction or via Hermite's method [5]. For instance, there exist 2 stationary points

$$\mathfrak{S}_1 \approx (2.666216,\ 1.234430),\ \ \mathfrak{S}_2 \approx (2.744834,\ 3.244859)$$

for the case $m_2 = 2, m_3 = 2$, and 4 stationary points

$$\mathfrak{S}_1 \approx (1.941246,\ 2.552370), \mathfrak{S}_2 \approx (2.655622,\ 1.638871),\ \ \mathfrak{S}_3 \approx (3.330794,\ \ 2.826444),$$

and

$$\mathfrak{N} \approx (2.552939,\ 2.271691)$$

for the case $m_2 = 2, m_3 = 4$. Hereinafter we denote by $\mathfrak{S}$ the saddle-type stationary point and by $\mathfrak{N}$ the minimum point.

In order to find the boundary in the parameter (m_2, m_3)-plane between the two distinct qualitative pictures — i.e. two vs. four stationary points — let us find the *discriminant curve*. Any pair of bifurcation values corresponds to the case when at least one stationary point becomes degenerate, i.e. if these bifurcation values are perturbed somehow, this stationary point either splits into (at least) two ordinary, nondegenerate stationary points or disappears at all. Therefore these bifurcation values for parameters can be found from the condition of changing the number of real solutions of the system (9). Hence, the bifurcation values correspond to the case when the multiple zero for the polynomial (10) appears. This condition is equivalent to vanishing of the discriminant

$$\mathcal{D}_y(\mathcal{Y}) = \mathcal{R}_y(\mathcal{Y},\mathcal{Y}'_y)\ . \tag{11}$$

[1] On excluding an extraneous factor.

This is a huge polynomial, which can be factored over $\mathbb{Z}$ as

$$\Xi^2(m_2,m_3)\Psi(m_2,m_3) \quad \text{with} \quad \deg \Xi = 444, \deg \Psi = 48 \ .$$

The condition $\Xi(m_2,m_3)=0$ corresponds to the case where the multiple zero for (11) appears due to the coincidence of the values of y-components for a pair of distinct solutions of the system (9) while the condition

$$\Psi(m_2,m_3)=0 \tag{12}$$

corresponds in the (m_2,m_3)-plane to the case of appearance of at least one degenerate zero for (9). The polynomial $\Psi(m_2,m_3)$ is an even one with respect to the involved parameters, and its expansion in powers of these parameters contains 325 terms. The complete expression can be found in [12], while here we demonstrate just only its terms of the highest and the lowest orders:

$$\begin{aligned}\Psi(m_2,m_3) = 3^{36}(64\,m_3^2+192\,m_2m_3+169\,m_2^2)^5(64\,m_3^2-192\,m_2m_3+169\,m_2^2)^5\times \\ (28561\,m_2^4+19968\,m_2^2m_3^2+4096\,m_3^4)^7 \\ +\dots \\ +2^2\cdot 3^{31}\cdot 17^{40}(5545037166327\,m_2^4-161882110764644\,m_2^2m_3^2+1656772227072\,m_3^4) \\ +2^3\cdot 3^{36}\cdot 17^{44}(51827\,m_2^2+28112\,m_3^2)+3^{36}\cdot 17^{48} \ .\end{aligned}$$

Drawing out the 48th order algebraic curve (12) is looking like an impossible mission. Fortunately[2], we have succeeded to do this. Since we are dealing with positive values of parameters, in Fig. 1 we present only the 4 “arrowhead”-looking branches of the curve lying in the first quadrant of the (m_2,m_3)-plane. Only one of these branches is the true bifurcation curve — the one presented in Fig. 2. The coordinates of its “vertices”, i.e. singular points, are as follows:

$$M_1 \approx (1.812918, 2.575996), M_2 \approx (2.886962, 5.667175),$$

$$M_3 \approx (1.236728, 3.556856) \ .$$

The values of the parameters lying in the interior of this branch, i.e. satisfying the inequality $\Psi(m_2,m_3)<0$, correspond to the case of existence of (precisely) 4 stationary points for the potential, while those lying outside this branch — to that of 2 stationary points.

It is worth mentioning, that for the case of existence of 4 stationary points, one of them is necessarily the minimum point for the potential, or, in terms of the differential equation theory, it gives the (asymptotically) stable equilibrium position for the vector field generated by the given configuration of charges. The condition $\Psi(m_2,m_3)<0$ can therefore be tackled as a *necessary* one for the existence of the minimum (stable) point for the potential function; it should

[2] Thanks to the open-source mathematical software system **Sage**, http://www.sagemath.org.

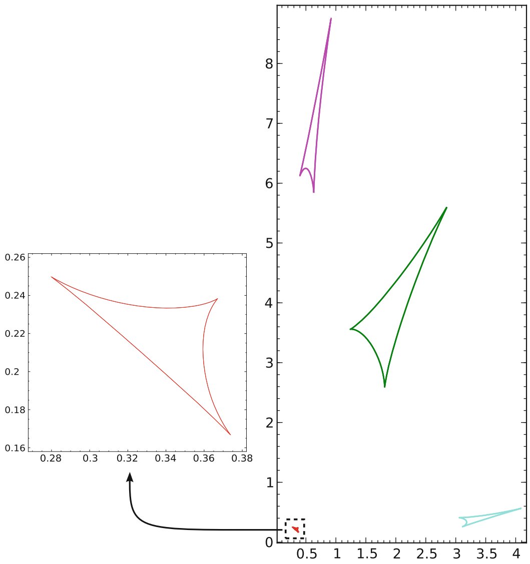

Fig. 1. Discriminant curve

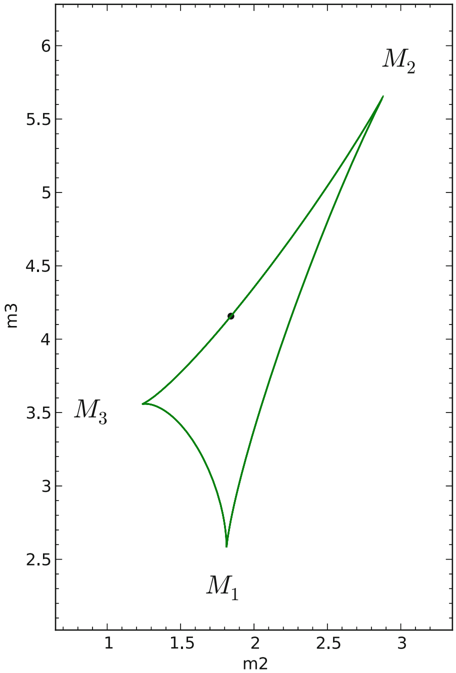

Fig. 2. Stability domain in parameter plane

be considered as a tight condition for stability in the set of semi-algebraic conditions. It can be transformed into the *necessary and sufficient* conditions via supplementing it with the system of linear inequalities providing the interior of the triangle $M_1M_2M_3$. As for the simple *sufficient* conditions for the existence of a stable point, one can obtain them in the form of system of linear inequalities providing a triangle $N_1N_2N_3$ lying inside the stability domain. For instance, one of such a triangle has the vertices $N_1 = (1.8, 3), N_2 = (2.6, 5.1)$ and $N_3 = (1.4, 3.6)$.

The choice of the point (m_2, m_3) right on the discriminant curve corresponds to the case when stationary point set contains precisely 3 points. For instance, for

$$(m_2, m_3) \approx (1.842860, 4.157140)$$

(this point is marked in Fig. 2) the stationary point set consists of

$$\hat{\mathfrak{S}}_1 \approx (2.691693, 1.930238), \tag{13}$$

$$\mathfrak{S}_2 \approx (1.821563, 2.558877), \mathfrak{S}_3 \approx (3.374990, 2.739157) \, ,$$

with $\hat{\mathfrak{S}}_1$ being a degenerate stationary point of the saddle-node type. □

The next challenging problem is that of localization of the stationary points for the potential (4). The determinant of the Hessian matrix $H(F)$ computed for this function at a stationary point $P_* = (x_*, y_*)$ is positive if P_* is a minimum point and negative if it is of saddle type. Therefore the inequality $\det H(F) > 0$ specifies a sharp (i.e., unimprovable) condition under which the minimum point exists. One may first compute $H(F)$ for arbitrary point $P = (x, y)$, and after that replace in it the parameters m_1, m_2 and m_3 with the aid of the ratio (7). The resulting condition will depend only on the variables x and y:

Theorem 2. *If there exists a minimum point for the potential (4), then it is located in the domain* $\mathbb{M}$ *of the triangle* $P_1P_2P_3$ *defined by the inequality*

$$\varPhi(x, y) > \frac{2}{9} S^2 \ . \tag{14}$$

Here S *is defined by (3) while*

$$\varPhi(x, y) = \frac{S_1(x, y) S_2(x, y) S_3(x, y)}{|PP_1|^2 |PP_2|^2 |PP_3|^2} C(x, y) \tag{15}$$

where

$$C(x, y) = S_1(x, y)|PP_1|^2 + S_2(x, y)|PP_2|^2 + S_3(x, y)|PP_3|^2$$
$$\equiv \left| \begin{array}{cccc} 1 & 1 & 1 & 1 \\ x & x_1 & x_2 & x_3 \\ y & y_1 & y_2 & y_3 \\ x^2 + y^2 & x_1^2 + y_1^2 & x_2^2 + y_2^2 & x_3^2 + y_3^2 \end{array} \right|$$

and $\{S_j(x,y)\}_{j=1}^3$ *are defined by (6). Conversely, any point* $P_* = (x_*, y_*)$ *lying in* $\mathbb{M}$ *is a minimum point for the potential (4) with any specialization of charges* m_1, m_2, m_3 *proportional to the values*

$$m_1^* = S_1(x_*, y_*)|P_*P_1|^3, \ m_2^* = S_2(x_*, y_*)|P_*P_2|^3, \ m_3^* = S_3(x_*, y_*)|P_*P_3|^3 .$$

For the proof of this theorem we refer to [11].

Example 2. Find the domain $\mathbb{M}$ of possible minimum point location for the configuration from Example 1.

Solution. Here[3] $S = 20$ and

$$\Phi(x,y) = \frac{16(28-5\,x-3\,y)(5\,x-y-4)(y-1)(-52+30\,x+32\,y-5\,x^2-5\,y^2)}{((x-1)^2+(y-1)^2)((x-5)^2+(y-1)^2)((x-2)^2+(y-6)^2)} .$$

The domain $\mathbb{M}$ is located inside the oval of the 6th order algebraic curve displayed in Fig. 3.

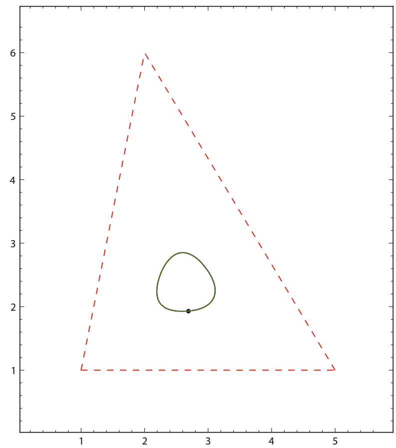

Fig. 3. Domain of possible minimum point location

One might expect that the point chosen on the curve corresponds to such values of parameters m_2 and m_3 that provide the degeneracy property of a stationary point for the potential. This is indeed the case: the one-to-one correspondence can be established between the points on this curve and those on the curve (12). For instance, the point marked on the curve in Fig. 3 is a degenerate stationary one for the potential (4) with $m_1 = 1, m_2 \approx 1.842860, m_3 \approx 4.157140$; its coordinates (13) have appeared in solution of Example 1. □

To conclude the treatment of the three point case, let us consider the configuration of equal charges with one of their placement variable .

[3] In the article [11], the expressions for S and Φ are provided with typos.

Example 3. Let $P_1 = (0,0), P_2 = (1,0), P_3 = (x_3, y_3)$ and $m_1 = m_2 = m_3 = 1$. Analyse the structure of the set of stationary points of the function (4).

Solution. The idea is similar to the solution of Example 1. We skip the intermediate computations and present the final result: the discriminant curve in (x_3, y_3)-parameter plane is given implicitly as

$$\Theta(x_3, y_3) = 0\ . \tag{16}$$

Here $\Theta(x_3, y_3)$ is the 76th order polynomial with respect to both coordinates. Its complete expression can be found in [12], while here we demonstrate only the terms of the highest and the lowest orders:

$$\Theta(x_3, y_3) =$$

$$2^{36}\cdot 3^{42}(9\,x_3^2+8\,y_3^2)^6(x_3^2+y_3^2)^{32}-2^{37}\cdot 3^{42}\,x_3(155\,y_3^2+171\,x_3^2)(9\,x_3^2+8\,y_3^2)^5(x_3^2+y_3^2)^{31}$$

$$+\dots$$

$$-2^{37}\cdot 3^{42}\,x_3(155\,y_3^2+171\,x_3^2)(9\,x_3^2+8\,y_3^2)^5(x_3^2+y_3^2)+2^{36}\cdot 3^{42}(9\,x_3^2+8\,y_3^2)^6(x_3^2+y_3^2)\ .$$

(There are no terms of degree lesser than 14.) The curve (16) is symmetric with respect to the line $x_3 = 1/2$ and consists of two branches also symmetric with respect to the x_3-axis; one of these branches is displayed in Fig. 4. The coordinates of its singular points are as follows[4]:

$$Q_1 \approx (0.398295, 0.798718),\ \ Q_2 \approx (0.601705, 0.798718), Q_3 \approx (0.5, 1.002671)\,.$$

For the point P_3 placed inside this curve $(\Theta(x_3, y_3) < 0)$, the potential (4) possesses 4 stationary points, while for the point P_3 lying outside — just 2

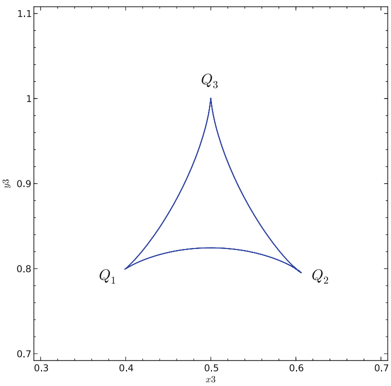

Fig. 4. The "upper" branch of the curve (16).

[4] Thus, one should not be misled by the visual illusion: the triangle $Q_1Q_2Q_3$ is not an equilateral one!

stationary points. Let us illuminate this statement considering the specialization $x_3 = 1/2$ which corresponds to the case of the isosceles triangle $P_1P_2P_3$. The corresponding stationary point set is symmetric with respect to the triangle median line $x_3 = 1/2$. One has

$$\begin{aligned}\Theta(1/2, y_3) \equiv & \frac{1}{2^{48}}(65536\, y_3^8 + 16384\, y_3^6 - 13824\, y_3^4 - 15552\, y_3^2 - 2187) \\ & \times (16384\, y_3^8 + 4096\, y_3^6 - 6912\, y_3^4 - 11664\, y_3^2 - 2187)^3 T^2(y_3) \ .\end{aligned}$$

Here $T(y_3)$ denotes an even polynomial of the degree 22 without real roots, i.e. $T(y_3) \neq 0$ for $y_3 \in \mathbb{R}$. The roots of the remained factors are the bifurcation values for y_3, and we restrict ourselves here only by positive values:

$$y_3^* \approx 0.824539 \text{ and } y_3^{**} \approx 1.002671.$$

For the choice $y_3 \in (y_3^*; y_3^{**})$ the stationary point set consists of 4 points. For instance, if $y_3 = 1$ then these points are

$$\mathfrak{S}_1 \approx (0.520962,\ 0.424850), \mathfrak{S}_2 \approx (0.479037,\ 0.424850), \mathfrak{S}_3 \approx (0.5,\ 0.075682),$$

$$\mathfrak{N} \approx (0.5,\ 0.423647).$$

For $y_3 \in (0; y_3^*) \cup (y_3^{**}; \infty)$ the stationary point set consists of 2 points. For instance, if $y_3 = 1/2$ then these points are:

$$\mathfrak{S}_1 \approx (0.267236,\ 0.219775),\ \mathfrak{S}_2 \approx (0.732763,\ 0.219775) \ ;$$

while for the choice $y_3 = 3/2$ both stationary points lie on the line $x_3 = 1/2$:

$$\mathfrak{S}_1 \approx (0.5,\ 0.029031),\ \mathfrak{S}_2 \approx (0.5,\ 0.783949). \qquad \square$$

3 Four Points

We now turn to the case of potential generated by configuration of 4 noncollinear charges $\{m_j\}_{j=1}^4$ placed at the points $\{P_j\}_{j=1}^4$

$$F(P) = \frac{m_1}{|PP_1|} + \frac{m_2}{|PP_2|} + \frac{m_3}{|PP_3|} + \frac{m_4}{|PP_4|} \ . \tag{17}$$

The idea of the proof of Theorem 1 can easily be extended to this case. System

$$\sum_{j=1}^4 \frac{m_j(x - x_j)}{|PP_j|^3} = 0,\ \sum_{j=1}^4 \frac{m_j(y - y_j)}{|PP_j|^3} = 0 \tag{18}$$

can be resolved — as a linear one — with respect to m_1 and m_2:

$$\left\{\begin{array}{l} m_1S_3/|PP_1|^3 = m_3S_1/|PP_3|^3 + m_4S_4/|PP_4|^3, \\ m_2S_3/|PP_2|^3 = m_3S_2/|PP_3|^3 + m_4S_5/|PP_4|^3. \end{array}\right. \tag{19}$$

Here S_1, S_2, and S_3 are defined by (6) while

$$S_4(x,y) = \left| \begin{array}{ccc} 1 & 1 & 1 \\ x & x_2 & x_4 \\ y & y_2 & y_4 \end{array} \right| , \ S_5(x,y) = \left| \begin{array}{ccc} 1 & 1 & 1 \\ x_1 & x & x_4 \\ y_1 & y & y_4 \end{array} \right| .$$

Next, the squaring procedure can be applied to both equations of the system (19). In two steps this results in the algebraic equations of degree 28 in x and y:

$$\tilde{F}_1(x,y,m_1,m_2,m_3,m_4) = 0, \ \tilde{F}_2(x,y,m_1,m_2,m_3,m_4) = 0 \ . \tag{20}$$

This result should be treated as an essential simplification in comparison with the squaring algorithm applied directly to system (18): the latter generates algebraic equations of the degree 72.

Example 4. Let $P_1 = (1,1), P_2 = (5,1), P_3 = (2,6), P_4 = (4,5)$. Analyse the structure of the set of stationary points of the function (17) for $m_1 = 1, m_2 = 2, m_3 = 4$ and for m_4 treated as parameter.

Solution. The given configuration of charges can be tackled as a perturbation of the 3 charges configuration from the solution of Example 1 with an extra charge placed at the position P_4.

We eliminate x variable from the system (20) with the aid of the resultant computation:

$$\mathcal{Y}(y,m_4) = \mathcal{R}_x(\tilde{F}_1, \tilde{F}_2) \ . \tag{21}$$

It can be factored over $\mathbb{Z}$ as

$$\mathcal{Y} \equiv W(m_4) G_1(y,m_4) G_2(y,m_4) (y-1)^{56} (y-5)^{16} (y-6)^{16} (4\,y^2 - 44\,y + 125)^{36} \ ;$$

here

$$W(m_4) \equiv m_4^{48} (m_4 - 5)^5 (m_4 + 5)^5$$

and, generically, $\deg_y G_1 = 180, \deg_y G_2 = 156$.

The y-components of the zeros of the system (20) are among the zeros of $G_2(y,m_4)$. Although we have succeed to compute this polynomial, we have failed to find its discriminant with respect to the variable y. Therefore we are not able to provide one with the bifurcation values set for the parameter m_4. We have established that one of these values lies within the interval $(4/15; 3/10)$, and when the parameter m_4 passes through this value while decreasing, the number of stationary point increases from 3 to 5. For instance, one obtains

$$\mathfrak{S}_1 \approx (1.952957, 2.176070), \ \mathfrak{S}_2 \approx (4.239198, \ 2.677284), \ \mathfrak{S}_3 \approx (3.154287, \ 5.396890)$$

for $m_4 = 2$ and

$$\mathfrak{S}_1 \approx (1.988731, 2.474302), \ \mathfrak{S}_2 \approx (2.603988, 1.852183), \ \mathfrak{S}_3 \approx (3.593059, 2.883524),$$

$$\mathfrak{S}_4 \approx (3.566307, 5.178565), \mathfrak{N} \approx (2.560190, 2.031979)$$

for $m_4 = 4/15$. $\square$

In order to confirm Maxwell's conjecture for the case of $K = 4$ charges, we have generated about thirty variants of their configuration. The number of stationary points never exceeds 7.

Example 5. Find the stationary point set for the system of charges $m_1 = 1, m_2 = 3, m_3 = 1, m_4 = 3$ placed at $P_1 = (0,0), P_2 = (1/2,-1), P_3 = (1,0), P_4 = (1/2,1)$ respectively.

Solution. The quadrilateral $P_1P_2P_3P_4$ is a rhombus, therefore the considered configuration of charges possesses two axes of symmetry, namely the lines $x = 1/2$ and $y = 0$. The symmetry property is inherited by the set of stationary points of the generated potential:

$$\mathfrak{N}_{1,2} \approx (0.5, \pm 0.194213),$$

$$\mathfrak{S}_1 = (0.5, 0), \mathfrak{S}_{2,3} \approx (0.316723, \pm 0.323720), \mathfrak{S}_{4,5} \approx (0.683276, \pm 0.323720).$$

Hence, at present we are unable neither to disprove Maxwell's estimation nor to ascertain its attainability.

4 Conclusions

Analytical approach for the investigation of the set of stationary points for the Coulomb potential function $F(P) = \sum_{j=1}^{K} m_j / |PP_j|$ in $\mathbb{R}^2$ was developed. The efficiency of the approach was illuminated for system of $K = 3$ and $K = 4$ charges in case when the values of charges as well as their coordinates are *specialized*, i.e. for the case when numerical values for these parameters are assigned, one can establish the exact number of stationary points and localize them within the given tolerance in a finite number of elementary algebraic operations. Moreover, for the case of $K = 3$ charges, it is possible to find the bifurcation picture in the domain of parameter *variation*. In all the examples we have treated Maxwell's conjecture was confirmed.

The case of $K \geq 4$ points in the space remains for further investigation. On extrapolating the length of outputs in examples treated in the paper, one may predict the growth of complexity in computation and analysis of this problem. On the other hand, it should be mentioned that all the computations for the examples have been performed on a standard configuration personal computer. Thus, the planned usage of specialized software implemented on a high-performance computer looks promising.

The proposed approach for construction of bifurcation diagrams can be applied for establishing the stability or ultimate boundedness conditions in the parameter space for wide classes of dynamical systems, such as treated in [1].

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