Algebraic and Radical Potential Fields. Stability Domains in Coordinate and Parametric Space

Alexei Yu. Uteshev

Faculty of Applied Mathematics, St.Petersburg State University, 198504, Petergof, Universitetskij pr. 35 St.Petersburg, Russia

alexeiuteshev@gmail.com

Abstract. A dynamical system $dX/dt = F(X;\mathbf{A})$ is treated where $F(X;\mathbf{A})$ is a polynomial (or some general type of radical contained) function in the vectors of state variables $X \in \mathbb{R}^n$ and parameters $\mathbf{A} \in \mathbb{R}^m$. We are looking for *stability domains* in both spaces, i.e. (a) domain $\mathbb{P} \subset \mathbb{R}^m$ such that for any parameter vector specialization $\mathbf{A} \in \mathbb{P}$, there exists a stable equilibrium for the dynamical system, and (b) domain $\mathbb{S} \subset \mathbb{R}^n$ such that any point $X_* \in \mathbb{S}$ could be made a stable equilibrium by a suitable specialization of the parameter vector $\mathbf{A}$.

INTRODUCTION

Investigation of the qualitative behavior of trajectories of a dynamical system

$$dX/dt = \mathbf{F}(X;\mathbf{A}),\ X = (x_1,\dots,x_n) \in \mathbb{R}^n \tag{1}$$

including exploration of dependency of the picture on parameter vector $\mathbf{A} \in \mathbb{R}^m$ involved in the right-hand side of this system (*bifurcation analysis*) is a well-known of the ODE theory. with a lot of practical and theoretical applications. The present paper is focused onto a particular problem concerning the conditions for existence and location of an asymptotically stable equilibrium position for (1). Namely, we look for the *largest possible* domain $\mathbb{P}$ in parameter space $\mathbb{R}^m$ such that for any specialization of parameter vector $\mathbf{A}$ from this domain, there exists at least one asymptotically stable equilibrium position for (1). On the other hand, we want to localize such an equilibrium in the sense of location of the set $\mathbb{S}$ in coordinate space $\mathbb{R}^n$ where every point might be made an asymptotically stable equilibrium by a suitable specialization for $\mathbf{A} \in \mathbb{P}$. Every domain $\mathbb{S}$ or $\mathbb{P}$ will be hereinafter referred to as the **stability domain** in the corresponding spaces; this notion should not be confused with the *domain of asymptotical stability* of an equilibrium.

We will be focused onto the case where the entries of the vector functions $\mathbf{F} = (F_1,\dots,F_n)$ are either polynomials in both variables and parameters, i.e. $\{F_j \in \mathbb{R}[X,\mathbf{A}]\}_{j=1}^n$, or can be somehow "reduced" to the polynomial form in the meaning of the word to be clarified below. Even for the polynomial case, the stated problem of finding the domain $\mathbb{P}$ generally does not admit a semialgebraic solution. In [1] an example of the system (1) is presented with $n = 3$ and $\{F_j\}_{j=1}^3$ being homogeneous polynomials in X of the order 5 involving $m = 4$ parameters. For this system, the necessary and sufficient conditions for asymptotical stability of the equilibrium cannot be expressed in terms of a finite system of algebraic inequalities imposed on the parameters. It is worthy to note that the mentioned example corresponds to the case of the total degeneracy of the Jacobian

$$\mathcal{J}(\mathbf{F}) = \left[\partial F_j/\partial x_k\right]_{j,k=1}^n . \tag{2}$$

In the present article, we will restrict ourselves to the case where this Jacobian is a nondegenerate matrix for any $X \in \mathbb{R}^n, \mathbf{A} \in \mathbb{R}^m$ except for, probably, some manifold of codimension 1 in any of these spaces. First, we will be interested in the relative position of this manifold

$$\det \mathcal{J}(\mathbf{F}) = 0 \tag{3}$$

and the set of real solutions of the system providing the equilibrium points for (1), i.e.

$$F_1 = 0, \dots, F_n = 0 . \tag{4}$$

Due to polynomiality of the involved functions, the problem is then reduced to the general problem of localization of real solutions of system of algebraic equations, i.e. establishing the exact number of real zeros of (4) and those satisfying an additional condition in the form of an algebraic equation or inequality. This problem can be resolved in the ideology of *symbolic computations*, i.e. via a purely algebraic algorithm consisting of a finite number of elementary algebraic operations on the coefficients of polynomials.

We will restrict our treatment of the problem to the case of bivariate systems containing two parameters (i.e. $m = n = 2$) due to its simplicity in visualization and in presentation of algebraic backgrounds.

ALGEBRAIC PRELIMINARIES

The basic tools for the treatment of bifurcation scenario for the set of zeros of an algebraic equation system are the **resultant** and the **discriminant** [2]. The resultant permits one to **eliminate** variables from the system (4), i.e. to obtain a univariate polynomial with the set of zeros containing any coordinate of a zero for the system (4). For instance, for the case of bivariate system

$$F_1(x, y) = 0,\ F_2(x, y) = 0 \quad \text{with } \{F_1, F_2\} \subset \mathbb{R}[x, y] , \tag{5}$$

the resultant $\mathcal{X}(x) := \mathcal{R}_y(F_1, F_2)$ of the polynomials F_1, F_2 treated with respect to the variable y, is a univariate polynomial in x such that any zero of $\mathcal{X}(x)$ is the x-component for at least one zero for the system (5). Provided that all the zeros of $\mathcal{X}(x)$ are simple, the system can be reduced to an equivalent one in the form

$$\mathcal{X}(x) = 0,\ y - G(x) = 0 \quad \text{with } \{\mathcal{X}, G\} \subset \mathbb{R}[x], \deg G < \deg \mathcal{X} . \tag{6}$$

Computational procedure consists of a finite number of elementary algebraic operations on the coefficients of F_1 and F_2. Generically, the number of real zeros for the system (5) (i.e. its solutions $(x_*, y_*) \in \mathbb{R}^2$) coincides with those of $\mathcal{X}(x)$, and the analysis of the former can be reduced to that of the latter. An exception to this claim might take place when some real zero for $\mathcal{X}(x)$ happens to be multiple. Such a zero might appear not only as a result of collision of two (or more) real zeros of the system (5) (which additionally results in the vanishement of the determinant of the Jacobian $\mathcal{J}(F_1, F_2)$) but also as an occasion that imaginary in the y-component zeros possess the common real x-component.

Computation of the exact number of real zeros for a univariate polynomial

$$F(x) := A_0 x^N + A_1 x^{N-1} + \ldots + A_N,\ A_0 \neq 0, N \geq 2$$

with numerical specialization of coefficients can be organized via the Sturm series construction or, alternatively, with the aid of Hermite's method [3]. Both methods can be viewed as the procedures for computation the discriminant of the polynomial

$$\mathcal{D}_x(F) := \mathfrak{D}(A_0, A_1, \ldots, A_N) ,$$

i.e. a polynomial expression whose vanishement provides the necessary and sufficient conditions for the existence of multiple zeros for $F(x)$. Treating the coefficients of the polynomial as the point $(A_0, A_1, \ldots, A_N)$ in $\mathbb{R}^{N+1}$, the equation $\mathfrak{D} = 0$ defines in this space the manifold separating the domains corresponding to polynomials $F(x)$ with distinct numbers of real zeros. In classical Algebra textbooks this manifold is known as the **discriminant surface**. Assume now the coefficients $\{A_j\}_{j=0}^N$ to be polynomial functions of some parameters. Then the *bifurcation* values for the latter, i.e. those values in any neighborhood of which there exist the parameter specialization corresponding to polynomials with distinct numbers of real zeros, are located either in the discriminant surface (curve, in case of $m = 2$ parameters), or in the manifold $A_0 = 0$.

If an additional restriction is imposed onto the set of real zeros of the system (5) in the form of a real algebraic equation (or inequality) $K(x, y) = 0$ (or > 0) then one can reduce the problem of counting the number of the zeros to the univariate counterpart via the transfer to the system (6).

POLYNOMIAL POTENTIAL

We first treat the system (1) where $\mathbf{F}(X;\mathbf{A}) \equiv -\nabla_X f(X;\mathbf{A})$ for some polynomial $f(X;\mathbf{A}) \in \mathbb{R}[X;\mathbf{A}]$. Stability question is then equivalent to the one concerning the point of (local) minimum of f. For this case, the Jacobian (2) coincides with the Hessian $\mathcal{H}(f)$. Since this is a symmetric matrix, an ordinary (nondegenerate) stationary point for (1) can be either saddle or node, with the expected local bifurcation scenario being of the saddle-node type.

Example 1 *Locate stability domains in the parameter plane and the coordinate plane for*

$$f(x,y;\alpha,\beta) = -2\,x^3 - y^3 + \alpha x^2 + 6\,xy - 2\,x + \beta y\,.$$

Solution. Depending on parameter specializations, the resultant

$$\mathcal{X}(x) := \mathcal{R}_y(\partial f/\partial x, \partial f/\partial y) = 9\,x^4 - 6\,\alpha x^3 + (\alpha^2+6)x^2 - (2\alpha+18)x - 3\beta + 1$$

possesses from 0 to 4 real zeros. The regions in the parameter plane corresponding to different numbers of these zeros are separated by the branches of the discriminant curve $\Psi(\alpha,\beta) = 0$ with

$$\Psi(\alpha,\beta) := \mathcal{D}_x(\mathcal{X}) = [(\alpha+\beta)^2 + 12 - \alpha^2][(12-\alpha^2)^2 - 432(\alpha+\beta)] + 54(\alpha+\beta)(\alpha^2-12) - 3^9\,. \qquad (7)$$

They are displayed in Figure 1 (a). Choosing some test specializations for parameters within every such a domain, one can determine that existence of minimum point for f is guaranteed only for the domain $\mathbb{P}$ consisting of two separate subdomains colored in Figure 1 (a). Besides the point of minimum, the stationary point set for every corresponding polynomial contains also two saddle points and one point of maximum. Expression (7) coincides, up to a numerical

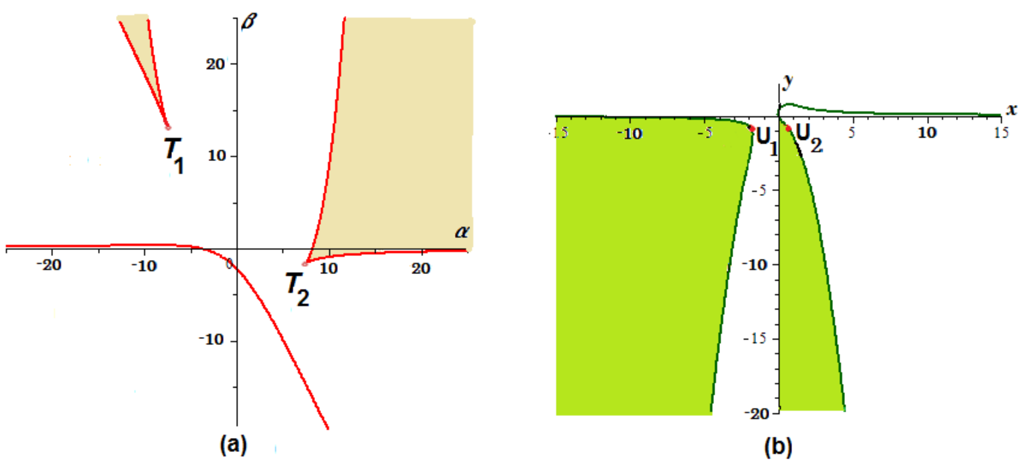

FIGURE 1. Example 1: stability domain in (a) parameter and (b) coordinate plane

factor, with the bivariate resultant of the polynomials $\partial f/\partial x, \partial f/\partial y$ and

$$\det \mathcal{H}(f) = 72\,xy - 12\,\alpha y - 36\,.$$

For specialization of parameters in $\mathbb{P}$, the necessary condition for a stationary point to be a minimum is the condition $\det \mathcal{H}(f) > 0$. If we resolve now the system $\partial f/\partial x = 0, \partial f/\partial y = 0$ with respect to the parameters α and β and substitute the result into this inequality, we obtain the inequality

$$12\Theta(x,y)/x > 0 \quad \text{where} \quad \Theta(x,y) := 3\,x^2 y + 3\,y^2 - 3\,x - y\,,$$

which describes the domain of stability $\mathbb{S}$ in the (x,y)-plane (Figure 1 (b)). There exists a one-to-one correspondence between the points in the discriminant curve $\Psi(\alpha,\beta) = 0$ and the curve $\Theta(x,y) = 0$: whilst the first one defines the parameter values α_*, β_* for which the corresponding potential possesses a degenerate stationary point, the second contains the coordinates of this point. If one variates continuously the point (α,β) tending it to (α_*,β_*) from the interior of the domain $\mathbb{P}$, then the degenerate stationary point in the coordinate plane results from the collision of two ordinary stationary points of the saddle and the node type. The exceptional cases correspond to the following singular points in the discriminant curve:

$$T_1 = (-\alpha_0, 9/\sqrt[3]{4} + \alpha_0) \approx (-7.407, 13.076),\ T_2 = (\alpha_0, 9/\sqrt[3]{4} - \alpha_0) \approx (7.407, -1.737) \quad \text{where } \alpha_0 = \sqrt{12 + 27\sqrt[3]{4}}\,.$$

The corresponding degenerate stationary points

$$U_1 = (-1/\sqrt[3]{4} - \alpha_0/6, -1/\sqrt[3]{4}) \approx (-1.864, -0.794),\ U_2 = (1/\sqrt[3]{4} + \alpha_0/6, -1/\sqrt[3]{4}) \approx (0.604, -0.794)$$

result from the collision of three ordinary stationary points.

RADICAL CONTAINING POTENTIALS

Let us now treat the potential

$$f(X; \{m_j\}_{j=1}^K, \{P_j\}_{j=1}^K, L) = \sum_{j=1}^{K} m_j |XP_j|^L\,. \tag{8}$$

Here $K \in \mathbb{N}, K \geq 3$, $\{m_j\}_{j=1}^K$ are assumed to be real positive numbers, $\{P_j\}_{j=1}^K$ are given points in $\mathbb{R}^n$, the exponent $L \in \mathbb{R}$ is nonzero while $|\cdot|$ stands for the Euclidean distance. Formula (8) includes, for various specializations of the exponent L, some known potentials [4, 5]. Of especial interest is the specialization $n = 3, L = -1$ corresponding to the Coulomb (or Newton) potential of the charges (or masses) $\{m_j\}_{j=1}^K$ fixed at the positions $\{P_j\}_{j=1}^K$ in the space. Although the equations for the stationary point coordinates of (8) are not of algebraic form when L is an odd number, such a form can be obtained via successive squaring. To optimize this procedure, one can utilize a trick which we illuminate for the case $n = 2, K = 3$, i.e. for the potential

$$f(x,y) = \sum_{j=1}^{3} m_j |XP_j|^L,\ \{P_j = (x_j, y_j)\}_{j=1}^3,\ X = (x,y)\,. \tag{9}$$

First resolve the system

$$\left\{ \begin{array}{rcl} \partial f/\partial x = & m_1|XP_1|^{L-2}(x-x_1) + m_2|XP_2|^{L-2}(x-x_2) + m_3|XP_3|^{L-2}(x-x_3) & = 0, \\ \partial f/\partial y = & m_1|XP_1|^{L-2}(y-y_1) + m_2|XP_2|^{L-2}(y-y_2) + m_3|XP_3|^{L-2}(y-y_3) & = 0 \end{array} \right. \tag{10}$$

with respect to m_1, m_2, m_3:

$$m_1 : m_2 : m_3 = |XP_1|^{2-L} S_1(x,y) : |XP_2|^{2-L} S_2(x,y) : |XP_3|^{2-L} S_3(x,y) \tag{11}$$

where

$$S_1(x,y) := \left| \begin{array}{ccc} 1 & 1 & 1 \\ x & x_2 & x_3 \\ y & y_2 & y_3 \end{array} \right|,\ S_2(x,y) := \left| \begin{array}{ccc} 1 & 1 & 1 \\ x_1 & x & x_3 \\ y_1 & y & y_3 \end{array} \right|,\ S_3(x,y) := \left| \begin{array}{ccc} 1 & 1 & 1 \\ x_1 & x_2 & x \\ y_1 & y_2 & y \end{array} \right|.$$

Then squaring the ratio (11) results in the system

$$m_2^2 S_1^2 |XP_1|^{2(2-L)} - m_1^2 S_2^2 |XP_2|^{2(2-L)} = 0,\ m_2^2 S_3^2 |XP_3|^{2(2-L)} - m_3^2 S_2^2 |XP_2|^{2(2-L)} = 0 \tag{12}$$

which is algebraic for any $L \in \mathbb{Z}$. Further analysis of real solutions of this system is similar to that outlined in the previous section. Stability domain in the coordinate plane is simpler in construction than its counterpart in the parameter space.

Theorem 1 *Let the noncollinear points P_1, P_2, P_3 be counted counterclockwise. Denote*

$$\Phi(x,y) := \frac{S_1(x,y) S_2(x,y) S_3(x,y)}{|PP_1|^2 |PP_2|^2 |PP_3|^2} \sum_{j=1}^{3} S_j(x,y) |XP_j|^2\,. \tag{13}$$

If $L \geq 1$ then any point P_* inside the triangle $P_1P_2P_3$ is a stable stationary point for the potential (9) where

$$\left\{ m_j = |P_*P_j|^{2-L} S_j(x_*, y_*) \right\}_{j=1}^{3}$$

Otherwise the boundary of the stability domain $\mathbb{S}$ in the coordinate plane is given by the equation

$$\Phi(x,y) = \frac{1-L}{(L-2)^2} S^2 \quad \textit{with} \quad S := \left| \begin{array}{ccc} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{array} \right| \equiv S_1(x,y) + S_2(x,y) + S_3(x,y)$$

(i.e., S equals the doubled square of the triangle).

Example 2 *Let* $P_1 = (1,1), P_2 = (5,1), P_3 = (2,6)$. *Find stability domains for the Coulomb potential*

$$1/|XP_1| + m_2/|XP_2| + m_3/|XP_3|. \tag{14}$$

Solution. Here $S = 20$ and

$$\Phi(x,y) = \frac{16(-5x - 3y + 28)(5x - y - 4)(y - 1)(-52 + 30x + 32y - 5x^2 - 5y^2)}{[(x-1)^2 + (y-1)^2][(x-5)^2 + (y-1)^2][(x-2)^2 + (y-6)^2]} \, .$$

Stability domain $\mathbb{S}$ is located inside the oval of algebraic curve of the order 6 displayed in Figure 2 (b).

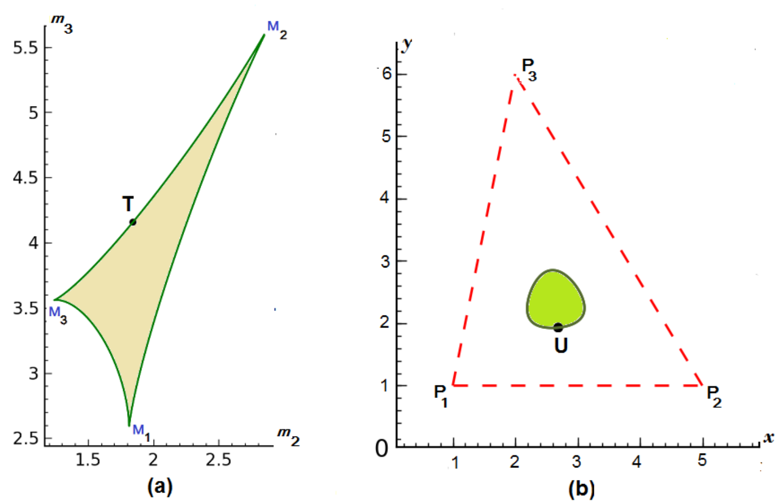

FIGURE 2. Example 2: stability domain in (a) parameter and (b) coordinate plane

The stability set $\mathbb{P}$ in the parameter plane is more complicated in construction. The resultant of the polynomials (13) treated with respect to x is (on excluding an extraneous factor) the polynomial $\mathcal{Y}(y, m_2, m_3)$, $\deg \mathcal{Y}_y = 34$. For any specialization of parameters m_2 and m_3, it is possible to find the exact number of its real zeros and to localize them in the ideology of symbolic computations, e.g., via the Sturm series construction or via Hermite's method [3]. For instance, the set of stationary points of the potential (15) consists of

- two saddles at $\approx (2.666,\ 1.234)$ and at $\approx (2.744,\ 3.244)$ for the case $m_2 = 2, m_3 = 2$;
- three saddles at $\approx$ $(1.941,\ 2.552)$, at $\approx$ $(2.655,\ 1.638)$, and at $\approx$ $(3.330,\ 2.826)$, and stable node at $\approx (2.552,\ 2.271)$ for the case $m_2 = 2, m_3 = 4$.

In order to find the boundary in the parameter (m_2, m_3)-plane between the two distinct qualitative picture — two vs. four stationary points — let us find the *discriminant curve*. Any pair of bifurcation values corresponds to the case when at least one stationary point becomes degenerate, i.e. if these bifurcation values are perturbed somehow, this stationary point either splits into (at least) two ordinary, nondegenerate stationary points or disappears at all. Therefore these bifurcation values for parameters can be found from the condition of changing the number of real solutions of the system (13). Hence, the bifurcation values correspond to the case when the multiple zero for the polynomial $\mathcal{Y}$ appears. This condition is equivalent to vanishing of the discriminant $\mathcal{D}_y(\mathcal{Y})$. This is a huge polynomial, which can be factored over $\mathbb{Z}$ as $\Xi^2(m_2, m_3)\Psi(m_2, m_3)$ with $\deg \Xi = 444, \deg \Psi = 48$. The condition $\Xi(m_2, m_3) = 0$ corresponds to the case where the multiple zero for $\mathcal{Y}$ appears due to the coincidence of the values of y-components for a pair of distinct zeros of the system (13) while the condition $\Psi(m_2, m_3) = 0$ corresponds to the case of appearance of at least one degenerate stationary point for the potential. The complete expansion for Ψ contains 325 terms with those of the highest order as follows

$$3^{36}(64\, m_3^2 + 192\, m_2 m_3 + 169\, m_2^2)^5 (64\, m_3^2 - 192\, m_2 m_3 + 169\, m_2^2)^5 (28561\, m_2^4 + 19968\, m_2^2 m_3^2 + 4096\, m_3^4)^7 .$$

The discriminant curve $\Psi(m_2, m_3) = 0$ consists of 4 branches in the first quadrant of the parameter plane; the one bounding the stability domain $\mathbb{P}$ is drawn in Figure 2 (a). The choice of the point (m_2, m_3) right in the discriminant curve corresponds to the case when two stationary points coincide. For instance, for $m_2 \approx 1.842, m_3 \approx 4.157$ (this point is marked T in Figure 2 (a)) the set of stationary points consists of the points $(1.821, 2.558), (3.374, 2.739)$ and $U \approx (2.691, 1.930)$, the latter being a degenerate stationary point of the saddle-node type (marked in Figure 2 (b)).

THE GENERAL CASE

Consider now the case of the system (1) not necessarily of the potential type. The local bifurcation scenario for the (dis)appearance of a stable equilibrium becomes more complicated due to possibility of existence of a bifurcation of changing stability for a focus, i.e. the one corresponding to the case of altering the sign of the real part of a pair of complex-conjugated eigenvalues of the Jacobian $\mathcal{J}(F_1, F_2)$. This imposes one extra restriction onto the zeros of the system $F_1(x, y) = 0, F_2(x, y) = 0$ which, in the considered bivariate case, can be represented with the aid of the trace of the Jacobian: $\mathbf{tr}\mathcal{J} < 0$.

Example 3 *Find stability domains for the system*

$$dx/dt = \alpha x + y + x^2 + \beta xy, \ dy/dt = -x + \alpha y + x^2 + y^2 .$$

Solution. For any specialization of parameters, the system possesses an equilibrium at $(0, 0)$ which is (un)stable for $\alpha < 0$ (for $\alpha > 0$). The discriminant curve for the saddle-node bifurcation is found in the similar manner to the previous examples, i.e. on elimination of the variables x, y from the equations $F_1 = 0, F_2 = 0$ and $\det \mathcal{J} = 0$:

$$\Psi_1(\alpha,\beta) := 4\,\alpha^4\beta^3 - 11\,\alpha^4\beta^2 + 2\,\alpha^3\beta^3 + \alpha^2\beta^4 + 10\,\alpha^4\beta - 10\,\alpha^3\beta^2 - 3\,\alpha^4 + 32\,\alpha^3\beta - 20\,\alpha^2\beta^2$$
$$-20\,\alpha^3 + 6\,\alpha^2\beta - 6\,\alpha\beta^2 - 4\beta^3 - 38\,\alpha^2 + 24\,\alpha\beta - 12\beta^2 - 24\,\alpha - 12\beta - 31 = 0 .$$

The unshadowed domain in Figure 3 (a) correspond to the systems possessing (precisely) 2 equilibria (saddle and focus), while shadowed domains to those possessing 4 equilibria:

- two saddles, stable focus and unstable node for the domain (I);
- tow saddles, unstable focus and stable node for the domain (II).

The structure of the equilibrium sets for the domain (III) is clarified below. Elimination of the variables from the equations $F_1 = 0, F_2 = 0$ and $\mathbf{tr}\mathcal{J} = 2\,x + (2 + \beta)y + 2\alpha = 0$ yields the curve

$$\Psi_2(\alpha,\beta) := \alpha(\alpha + 1)(\alpha^2\beta^3 - 4\,\alpha^2\beta^2 + 4\,\alpha^2\beta - 2\,\alpha\beta^2 - \beta^3 + 8\,\alpha\beta - 4\beta^2 - 8\,\alpha - 4\beta - 8) = 0 .$$

One of its branch (Figure 3 (b)), namely the line $\alpha = 0$, bounds the domain of stability of the focus $(0, 0)$. The line $\alpha = -1$ is a boundary of the domain defined by inequalities $\alpha < -1, \beta < -2, \Psi_1(\alpha,\beta) > 0$ where the parameter specializations furnishes an additional stable equilibrium for the system (Figure 3 (c)). This might be either focus or node:

- focus at $\approx (0.954, -0.021)$ for the case $\alpha = -2, \beta = -50$;
- degenerate node at $\approx (0.590, -0.0645)$ for the case $\alpha \approx -3.697, \beta = -50$;

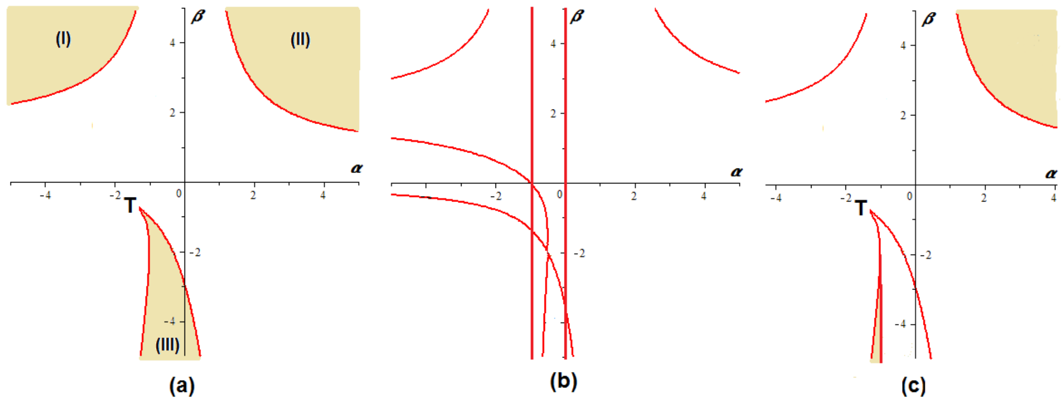

FIGURE 3. Example 3, parameter plane: (a) $\det \mathcal{J} = 0$; (b) $\mathbf{tr}\mathcal{J} = 0$; (c) domain of existence of an extra stable equilibrium

- node at $\approx (0.584, -0.0643)$ for the case $\alpha = -3.7, \beta = -50$.

Resolving now the equations $F_1 = 0, F_2 = 0$ with respect to α, β and substituting this into $\det \mathcal{J} > 0$, one obtains

$$(y^2 + x^2)\Theta_1(x, y)/(xy^2) > 0 \quad \text{where} \quad \Theta_1(x, y) := 2\,x^3 - x^2 y - 3\,x^2 + y^2 + x\,.$$

The curve $\Theta_1(x, y) = 0$ is not a boundary for the domain $\mathbb{S}$, this is a boundary for the domains where an extra stable equilibrium might locate aside from $(0, 0)$ (Figure 4; an extra branch of the curve, not displayed in that figure, is located in the domain $x > 6, y > 16$). For the parameter specializations in the domain (II) systems (1) this equilibrium

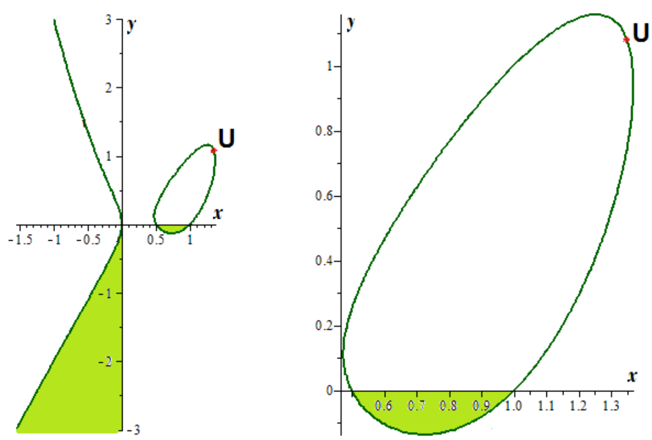

FIGURE 4. Example 3, coordinate plane: domain of existence of an extra stable equilibrium

is of the node type, while for specializations from the subdomain of the domain (III) defined by the condition $\alpha < -1$ this might be either node or focus. To distinguish these alternatives, one have to analyze an extra restriction imposed onto the zeros of the system $F_1(x, y) = 0, F_2(x, y) = 0$, namely the one related to the sign of the discriminant of the characteristic polynomial of the Jacobian: $\mathcal{D}_\lambda(\det[\lambda \mathbf{I} - \mathcal{J}])$ with $\mathbf{I}$ standing for the identity matrix. For the treated example, the corresponding discriminant curve $\Psi_3(\alpha, \beta) = 0$ is of the order 12, while its counterpart for the coordinate plane $\Theta_2(x, y) = 0$ is of the order 6. Both curves possess the branches lying very close to those of $\Psi_1(\alpha, \beta) = 0$ and $\Theta_1(x, y) = 0$ correspondingly.

The parameter specialization $\alpha \approx -1.515, \beta \approx -0.587$ (point T in Figure 3) furnishes a degenerate equilibrium for the system located at $U \approx (6.473, 16.988)$ (Figure 4) resulting from the collision of three ordinary equilibria.

Remark. An extension of the approach outlined in the present section to the multivariate case ($n \geq 3$) can be organized with the aid of analysis of the condition for existence of purely imaginary zeros of the characteristic polynomial of the Jacobian

$$\det[\lambda \mathbf{I} - \mathcal{J}] = \lambda^n + p_1\lambda^{n-1} + \ldots + p_n \in \mathbb{R}[\lambda, X, \mathbf{A}] \, .$$

This condition can be expressed in terms of the resultant of the polynomials

$$p_n + p_{n-2}\mu + p_{n-4}\mu^2 + \ldots \quad \text{and} \quad p_{n-1} + p_{n-3}\mu + p_{n-5}\mu^2 + \ldots$$

(i.e. the "even" and the "odd" parts of the characteristic polynomial) with respect to μ. Since this expression is algebraic with respect to X and $\mathbf{A}$, the elimination of variables or parameters from the system $F(X;\mathbf{A}) = \mathbb{O}$ accomplished with this condition can be performed in the manner similar to that demonstrated in the last example.

CONCLUSIONS

We have treated the problem of establishing the structure of sets of equilibria for the algebraic (and some "reducible to algebraic") dynamical systems. The approach is based on the reduction of the problem to the general problem of localization of real zeros of systems of multivariate algebraic equations. Classical Algebra algorithms for elimination of variables in such systems, as well as the analysis of the behavior of the discriminant manifold, allow us to separate, in both parameter space and in the space of variables, the domains with distinct structure of the sets of equilibria.

ACKNOWLEDGMENTS

This research was supported by the RFBR according to the project No **17-29-04288**.

REFERENCES

[1] V. I. Arnold, Funct. Anal. Appl. **4(3)**, 173–180 (1970).
[2] E. A. Kalinina and A. Yu. Uteshev, *Elimination Theory* (NII Khimii, St.Petersburg State University, St.Petersburg, 2005).
[3] A. Yu. Uteshev and S. G. Shulyak, Linear Algebra Appl. **177**, 49–88 (1992).
[4] A. Yu. Uteshev and M. V. Yashina, Lecture Notes Comput. Sc. **8136**, 412–426 (2013).
[5] A. Yu. Uteshev and M. V. Goncharova, Lecture Notes Comput. Sc. **9570**, 68–80 (2016).