ON THE EXISTENCE OF A POLYNOMIAL LYAPUNOV FUNCTION

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Consider the system of differential equations

$$dx_i/dt = f_i(x_1, \ldots, x_k)\,(i = 1, 2, \ldots, k). \tag{1}$$

Here $f_i(x_1, \ldots, x_k)\,(i = 1, \ldots, k)$ are homogeneous polynomials (forms) of a degree n. For brevity, we shall say that the system (1) is as. stable if an equilibrium $x_1 = \cdots = x_k = 0$ is asymptotically stable. We shall say also that the continuosly differentiable function $\mathcal{V}(x_1, \ldots, x_k)$ solves the stability problem for the system (1) if its derivative with respect to (1), i.e. the function

$$d\mathcal{V}/dt\,\Big|_{(1)} = \sum_{i=1}^{k} f_i \partial \mathcal{V}/\partial x_i$$

is sign-definite.

THEOREM 1. [1] *For the system (1) to be as. stable it is necessary and sufficient that there exist positive-definite homogeneous functions $\mathcal{V}$ and $\mathcal{W}$ of order m and $m+n-1$ respectively such that $d\mathcal{V}/dt\,\big|_{(1)} = -\mathcal{W}$.*

The evident question may be posed: is it possible to choose functions $\mathcal{V}$ and $\mathcal{W}$ quoted in Theorem 1 in the class of *forms*?

For $n = 1$ the positive answer is given by Lyapunov theorem **[2]**: if a linear system (1) is as. stable, then for any even m there exist forms $\mathcal{V}$ and $\mathcal{W}$ satisfying the Theorem 1.

Let us consider now the system

$$dx/dt = P(x, y),\ dy/dt = Q(x, y) \tag{2}$$

where P and Q are forms of a degree $n \geq 3$.

Definition 1. In the space of coefficients of the system (2) right-hand sides let us denote the set of as. stable systems by $AS(n)$, let $LS_m(n)$ be such a subset of $AS(n)$ that the as. stability problem for a system of this subset may be solved with the help of a form $\mathcal{V}$ of degree m.

The main result of the present note is the following

THEOREM 2. *For arbitrary $m : AS(3) \neq LS_m(3)$.*

In other words: for any m given there exists an as. stable system (2) where P and Q are forms of degree 3 such that the derivative of any form $\mathcal{V}$ of degree m with regard to this system will not be a negative-definite function.

LEMMA 1. *For the system*

$$dx/dt = \alpha x^3 - y^3,\ dy/dt = x^3 - \alpha y^3, \qquad 0 < \alpha < 1, \tag{3}$$

(0,0) is a center. For the system

$$dx/dt = (\alpha - \epsilon)x^3 - y^3,\ dy/dt = x^3 - \alpha y^3, \qquad 0 < \epsilon < \alpha < 1, \tag{4}$$

(0,0) is a stable focus.

The proof is given in [3], [4].

LEMMA 2. *For the system (3) the polynomial integrals of a degree $m - 2r$ exist only for a finite number of values of α.*

PROOF: Suppose that this integral exists. We get $\mathcal{V}(x,y) = A_0x^m + A_1x^{m-1}y + \cdots + A_my^m$, $A_0 \neq 0$ and $d\mathcal{V}/dt\,|_{(3)} = 0$. From the latter equality we obtain the homogeneous linear system of $(m+3)$ equations from $(m+1)$ unknowns $A_0, \ldots, A_m$. This system possesses the following property: if it has a solution $\mathbf{A}^{(1)} = (A_0, A_1, \ldots, A_m)$, then it necessarily has also a solution $\mathbf{A}^{(2)} = (A_m, A_{m-1}, \ldots, A_0)$ and, consequently, a symmetric one: $\mathbf{A}^{(3)} = \mathbf{A}^{(1)} + \mathbf{A}^{(2)} = (A_0 + A_m, A_1 + A_{m-1}, \ldots, A_m + A_0)$. $\mathbf{A}^{(3)} \not\equiv 0$ (otherwise, $A_m = -A_0$ and function $\mathcal{V}$ is (sign-) indefinite; this contradicts, however, the fact that $(0,0)$ is a center). Thus, if there exists a polynomial integral for the system (3), then there exists also a symmetric polynomial integral. Therefrom, we may consider that $A_i = A_{m-i}$.

Denote $u = x/y + y/x$. Then $\mathcal{V} = (xy)^r\mathcal{V}_1(u)$ and $\deg \mathcal{V}_1 = r$. By the condition $d\mathcal{V}/dt\,|_{(3)} \equiv 0$, we have the equation

$$\frac{d\ln \mathcal{V}_1}{du} = r\frac{u+\alpha}{u^2 - \alpha u - 2}. \tag{5}$$

Integrating it, we obtain

$$\mathcal{V}_1 = C(u - u_1)^{rB_1}(u - u_2)^{rB_2},$$
$$u_{1,2} = \frac{\alpha \pm \sqrt{\alpha^2 + 8}}{2},\ B_{1,2} = \frac{1}{2} \pm \frac{3\alpha}{2\sqrt{\alpha^2+8}},\ C = \text{const}.$$

$\mathcal{V}_1$ is a polynomial of degree r if and only if $0 < 3\alpha/\sqrt{\alpha^2+8} = p/r < 1$, where $p \in \mathbf{N}$ and $p+r$ is even. Thus, polynomial integral of a degree $m = 2r$ exists only if $\alpha = 2p\sqrt{2/(9r^2 - p^2)}$, $p < r$, $p + r$ is even. It does not exist for $r = 1, 2$; for $r = 3$ the polynomial integral exists if $\alpha = 1/\sqrt{10}$; for $r = 4$ the polynomial integral exists if $\alpha = \sqrt{8/35}$, etc. ∎

Definition 2. Every such a value of the α will be called a critical one.

By Lemma 2, the set of such parameters lying in the interval $(0,1)$ is at most countable.

LEMMA 3. *Consider a form of two variables of a given degree in the space of its coefficients. Then the set of semi-definite forms (of a fixed sign) complemented by 0-form is closed.*

PROOF: There exists a bijection between the set of sign-definite forms of two variables and the set of polynomials of one variable without real roots: $\mathcal{V}(x,y) \leftrightarrow \mathcal{V}(u,1)$. In the space of coefficients of a polynomial $\mathcal{V}(u,1)$ the transition from the set of absence of real roots into the set of their presence is possible only by intersection of the manifold consisting of the points which correspond to the polynomials with multiple roots. In addition, multiplicity of each multiple root is even if it happens to be real. Thus, the limit of an arbitrary convergent sequence of semi-definite forms of a fixed sign must be either semi-definite or 0-form. ∎

LEMMA 4. *Let $\mathcal{V}(x,y)$ and $\mathcal{W}(x,y)$ be the forms, where $\mathcal{W}$ is positively semi-definite and $d\mathcal{V}/dt\,\big|_{(2)} = -\mathcal{W}$. Then* $(0,0)$ *is not a center.*

PROOF: We shall use the following

THEOREM 3. [5, p. 28] *If there exists a positive definite function $\mathcal{V}(x_1,\dots,x_k)$ such that $d\mathcal{V}/dt\,\big|_{(1)} < 0$ outside M and $d\mathcal{V}/dt\,\big|_{(1)} \le 0$ on M, where M is a set which contains no entire trajectories apart from $O = (0,\dots,0)$, then O is as. stable.*

Suppose on the contrary that the assertion of Lemma 4 is false, i.e. $(0,0)$ is a center. Since $\mathcal{W}$ is a homogeneous function, the set $\mathcal{W} = 0$ may be either $(0,0)$ or the family of lines passing through $(0,0)$. Both possibilities contradict to Theorem 3. ∎

Let us prove now the main Theorem 2. Consider the set of as. stable systems (4). Let the assertion of the theorem be false, i.e. for every α and ϵ there exist two forms $\mathcal{V}_{\alpha,\epsilon}(x,y)$ and $\mathcal{W}_{\alpha,\epsilon}(x,y)$ of degrees m and $m+2$ respectively satisfying Theorem 1. Let us normalize the coefficients of the set of forms $\mathcal{V}_{\alpha,\epsilon}(x,y)$ by the condition $\sum_{i=0}^{m} A_i^2 = 1$ (this may be always done because for any positive constant C the form $C\mathcal{V}(x,y)$ solves also the stability problem for (4)). Then the coefficients B_i of the forms $\mathcal{W}_{\alpha\epsilon}(x,y)$ happen to be bounded:

$$\sum_{i=0}^{m+2} B_i^2 \le (4m)^2(m+3).$$

Let α be fixed. In the space of coefficients of forms $\mathcal{W}_{\alpha\epsilon}(x,y)$ consider the set $(B_0(\epsilon), B_1(\epsilon),\dots,B_{m+2}(\epsilon))\,(0 < \epsilon < \alpha)$. It is bounded. Thus, as $\epsilon \to +0$ it is possible to choose a convergent subsequence in this set. Its limit will be the point representing the form $\mathcal{W}_{\alpha,0}(x,y)$. By Lemma 3, this form may be either positive semi-definite or identically 0. Choose a non-critical α. $\mathcal{W}_{\alpha,0}(x,y) \not\equiv 0$ (otherwise $\mathcal{V}_{\alpha,0}(x,y)$ is a nontrivial integral of (3)). By Lemma 4, $\mathcal{W}_{\alpha,0}$ can not be also a positive semi-definite one. Contradiction completes the proof. ∎

COROLLARY. *For any* M : $AS(3) \ne \bigcup_{r=2}^{M} LS_r(3)$.

It is evident that if a form $\mathcal{V}$ solves the as. stability problem for the system (2), then any of its positive powers solves also that problem. Choose $m =$ *the least common multiple* $(2,\dots,M)$ and apply Theorem 2.

Example. The as. stability problem for the system (4) where $\alpha = 1,2$, $\epsilon = 1/4$ can not be solved by means of a quadratic form.

Solution. Let $\mathcal{V} = a_0x^2 + 2a_1xy + a_2y^2$, $a_0 > 0$, $a_2 > 0$, $a_1^2 < a_0a_2$.

1. Let $a_1 > 0$. $\mathcal{V} = a_1(A_0x^2 + 2xy + A_2y^2)$, where $A_i = a_i/a_1 > 0$ $(i = 0,2)$. Let $\mathcal{W}(x,y) = d\mathcal{V}/dt_{(4)}$. $\mathcal{W}(0,1) = 2a_1(-1 - A_2/2) < 0$, $\mathcal{W}(1,0) = 2a_1(1 + A_0/4) > 0$, thus, $\mathcal{W}$ is indefinite.

2. Let $a_1 < 0$. $\mathcal{V} = -a_1(A_0x^2 - 2xy + A_2y^2)$, where $A_i = -a_i/a_1 > 0$, $(i = 0,2)$. $\mathcal{W}(0,1) < 0$ if $A_2 > 2$, $\mathcal{W}(\sqrt[3]{4},1) < 0$ if $A_2 < \sqrt[3]{4}$, conditions on A_2 are inconsistent.

3. Let $a_1 = 0$. $\mathcal{W}(1,0) > 0$, $\mathcal{W}(1,0) < 0$, therefore Lyapunov function $\mathcal{V}(x,y)$ can not ve found in the set of quadratic forms. ∎

Two interesting problems should be noted:

1. $AS(n) = \bigcup_{r=2}^{\infty} LS_r(n)$?
2. $LS_{2m}(n) \subseteq LS_{2m+2}(n)$?

The author takes this opportunity to thank V. I. Zubov for suggesting the problem and for directing the work.

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