



Analytics of the Multifacility Weber Problem

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Abstract. For the Weber problem of construction of the minimal cost planar weighted network connecting four terminals with two extra facilities, the solution by radicals is proposed. The conditions for existence of the network in the assumed topology and the explicit formulae for coordinates of the facilities are presented. It is shown that the bifacility network is less costly than the unifacility one. Extension of the results to the general Weber problem is also discussed.

Keywords: Multifacility location problem · Weber problem · Nonlinear optimization

1 Introduction

The classical *Weber* or *generalized Fermat-Torricelli problem* is stated as that of finding the point (facility, junction) $W = (x_*, y_*)$ that minimizes the sum of weighted distances from itself to $n \geq 3$ fixed points (terminals) $\{P_j = (x_j, y_j)\}_{j=1}^n$ in the Euclidean plane:

$$\min_{W \in \mathbb{R}^2} \sum_{j=1}^n m_j |WP_j|. \quad (1)$$

Hereinafter $|\cdot|$ stands for the Euclidean distance and the *weights* $\{m_j\}_{j=1}^n$ are assumed to be positive real numbers.

The treatment of the problem in the case $n = 3$ terminals was first undertaken in 1872 by Launhardt [4] whose interest stemmed from the evident relation to the Economic Geography problem nowadays known as *Optimal Facility Location*. For instance, one can be interested in minimizing transportation costs for a plant manufacturing one ton of the final product from $\{m_j\}_{j=1}^n$ tons of distinct raw materials located at corresponding $\{P_j\}_{j=1}^n$.

Further development of the problem was carried out in 1909 by Alfred Weber [10]. First, he suggested a different economic interpretation for the three-terminal problem. Let P_3 be a place of consumption of m_3 tons of a product produced

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from two different types of raw materials: m_1 tons of the first type located at P_1 and m_2 tons of the second type located at P_2 , let $m_3 < m_1 + m_2$. Where is the optimal location of the production? In the course of the economic background, Weber formulated the following extension of the problem to the case of four terminals¹.

“Let us take a simple case, an enterprise with three material deposits and one which is capable of being split, technologically speaking, into two stages. In the first stage two materials are combined into a half-finished product; in the second stage this half-finished product is combined with the third material into the final product. . . Let us suppose that possible location of the split production would be in W_1 and W_2 ; W_1 for the first stage and W_2 for the second stage. What will be the result if the splitting occurs?” [10].

Mathematically the stated problem can be formulated as that of finding the points $W_1 = (x_*, y_*)$ and $W_2 = (x_{**}, y_{**})$ which yield

$$\min_{\{W_1, W_2\} \subset \mathbb{R}^2} F(W_1, W_2) \quad \text{where}$$

$$F(W_1, W_2) = m_1|W_1P_1| + m_2|W_1P_2| + m_3|W_2P_3| + m_4|W_2P_4| + m|W_1W_2| \quad (2)$$

and the weights $\{m_j\}_{j=1}^4, m$ are treated as given positive real numbers.

The general *Multifacility Weber problem* is stated as that of location of the given number $\ell \geq 2$ of the facility points (or, simply, facilities) $\{W_i\}_{i=1}^\ell$ in \mathbb{R}^d connected to the terminals $\{P_j\}_{j=1}^n \subset \mathbb{R}^d$ that solve the optimization problem

$$\min_{\{W_1, \dots, W_\ell\} \subset \mathbb{R}^d} \left\{ \sum_{j=1}^n \sum_{i=1}^\ell m_{ij}|W_iP_j| + \sum_{k=1}^\ell \sum_{i=k+1}^{\ell-1} \tilde{m}_{ik}|W_iW_k| \right\}; \quad (3)$$

here some of the weights m_{ij} and \tilde{m}_{ik} might be zero. We will refer to this value as to the **minimal cost of the network**. This problem can be considered as a natural generalization of the celebrated *Steiner minimal tree problem* aimed at construction of the network of minimal length linking the given terminals.

Dozens of papers are devoted to the Weber problem, its ramifications and applications; we refer to [3, 5, 11] for the reviews. The majority of them are concerned with the problem statement where the objective function (3) is free of the inter-facilities connections. This problem is known as the *Multisource Weber problem* or the ℓ -*median problem*. The present paper is focused on a solution to the Multifacility Weber problem. The mainstream approach in the treatment of this nonlinear optimization problem is the one based on reducing it to an appropriate iterative numerical procedure. For instance, the unifacility version of the problem (1) can be resolved via the modified Weiszfeld algorithm. The main obstacle of this approach consists in the fact that the objective (or cost) function of the Weber problem is non-differentiable at terminal points, and the

¹ In the quote we change the original notation of the points.

iterative procedure might diverge if any of the facilities happens to lie close to a terminal (or, in case of the multifacility problem, if two facilities are about to collide).

The present paper is devoted to an alternative approach for the problem, namely an analytical one. We are looking for the conditions for existence of the network and the explicit expressions for the facility coordinates in terms of the problem parameters, i.e. terminal coordinates and weights. This approach has been originated in the recent paper [6] where the unifacility Weber problem for three terminals had been solved *by radicals*. Within the framework of this approach, we will focus here on solution to the planar multifacility Weber problem for the case of $n = 4$ terminals and $\ell = 2$ facilities (i.e. the problem (2)), and also for the case of $n = 5$ terminals and $\ell = 3$ facilities.

Our analytical treatment stems from the geometric solution to the problem originated by Georg Pick and published in the Mathematical Appendix of Weber's book [10]. Nevertheless, Pick did not provide any proof of validity for his algorithm. In the conference paper [8] the present authors have announced without a proof the claim that the Weber problem (2) is solvable by radicals. In a simplified version (and with an extra assumption missed in [8]), this statement is now proved in Sect. 3. In addition, the conditions for the existence of the desired configuration of the network are provided.

In the case of the problem involving variable parameters, analytics provides one with a unique opportunity to evaluate their influence on the solution. In particular, it gives the means to determine the *bifurcation values* for these parameters, i.e. those responsible for the degeneracy of the network topology. We discuss these issues in Sect. 4 via investigation of the facilities dynamics under variation of the terminals location or the value of the involved weights. One may imagine a relevant economic optimization problem with a trawler fishing in the ocean and a floating fish processing facility drifting in anticipation of the catch transferred to it. We also prove here that, in the case of existence, the optimal bifacility network has its cost lower than the unifacility one.

In Sect. 5, we briefly discuss an opportunity for extension of the results to the case of $n \geq 5$ terminals and $\ell \geq 3$ facilities. This extension is based on the reduction of the problem to a similar one with $n - 1$ terminals and $\ell - 1$ facilities via a replacement of a pair of terminals by a suitable auxiliary *phantom* terminal. This trick is just a counterpart of the one utilized in Melzak's algorithm for Steiner tree construction.

2 Unifacility Configuration

Analytical solution for the three-terminal problem, i.e. for finding

$$\min_{W \in \mathbb{R}^2} (m_1 |WP_1| + m_2 |WP_2| + m_3 |WP_3|) \quad (4)$$

is given in [6]. In the present section we assume the vertices $P_j = (x_j, y_j)$ of the triangle $P_1P_2P_3$ be counted counterclockwise. Thus, the value

$$\mathfrak{S} = x_1y_2 + x_2y_3 + x_3y_1 - x_1y_3 - x_3y_2 - x_2y_1$$

is two times the area of this triangle. Denote

$$r_{ij} = |P_i P_j| = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} \quad \text{for } \{i, j\} \subset \{1, 2, 3\},$$

by $\alpha_1, \alpha_2, \alpha_3$ the angles of the triangle $P_1 P_2 P_3$, while by $\beta_1, \beta_2, \beta_3$ the angles of the so-called weight triangle formed by the triple of weights m_1, m_2, m_3 . The value $\sqrt{k}/4$ where

$$k = (m_1 + m_2 + m_3)(-m_1 + m_2 + m_3)(m_1 - m_2 + m_3)(m_1 + m_2 - m_3) \quad (5)$$

is the Heron formula for the area of the weight triangle.

Theorem 1. *The necessary and sufficient condition for the existence of solution to the problem (4) is that of the following system of inequalities $\{\cos \alpha_j + \cos \beta_j > 0\}_{j=1}^3$. Under this condition, the coordinates of the optimal facility $W = (x_*, y_*)$ are given by the formulae*

$$x_* = \frac{K_1 K_2 K_3}{2\mathfrak{S}\sqrt{k}d} \left(\frac{x_1}{K_1} + \frac{x_2}{K_2} + \frac{x_3}{K_3} \right), \quad y_* = \frac{K_1 K_2 K_3}{2\mathfrak{S}\sqrt{k}d} \left(\frac{y_1}{K_1} + \frac{y_2}{K_2} + \frac{y_3}{K_3} \right)$$

with the cost of the optimal network $\mathfrak{C} = \sqrt{d}$. Here

$$d = \frac{1}{\sqrt{k}}(m_1^2 K_1 + m_2^2 K_2 + m_3^2 K_3)$$

$$= |\mathfrak{S}|\sqrt{k} + \frac{1}{2} [m_1^2(r_{12}^2 + r_{13}^2 - r_{23}^2) + m_2^2(r_{23}^2 + r_{12}^2 - r_{13}^2) + m_3^2(r_{13}^2 + r_{23}^2 - r_{12}^2)],$$

and

$$\begin{cases} K_1 = (r_{12}^2 + r_{13}^2 - r_{23}^2)\sqrt{k}/2 + (m_2^2 + m_3^2 - m_1^2)\mathfrak{S}, \\ K_2 = (r_{23}^2 + r_{12}^2 - r_{13}^2)\sqrt{k}/2 + (m_1^2 + m_3^2 - m_2^2)\mathfrak{S}, \\ K_3 = (r_{13}^2 + r_{23}^2 - r_{12}^2)\sqrt{k}/2 + (m_1^2 + m_2^2 - m_3^2)\mathfrak{S}. \end{cases}$$

The proof consists in formal verification of the equalities

$$m_1 \frac{x_* - x_1}{|WP_1|} + m_2 \frac{x_* - x_2}{|WP_2|} + m_3 \frac{x_* - x_3}{|WP_3|} = 0, \quad (6)$$

$$m_1 \frac{y_* - y_1}{|WP_1|} + m_2 \frac{y_* - y_2}{|WP_2|} + m_3 \frac{y_* - y_3}{|WP_3|} = 0, \quad (7)$$

providing the stationary points of the objective function $\sum_{j=1}^3 m_j |P_j W|$.

The theorem states that the three-terminal Weber problem is solvable by radicals. Geometric meaning of the constants appeared in this theorem is as follows: $\frac{1}{2}|\mathfrak{S}|$ equals the area of the triangle $P_1 P_2 P_3$ while $\frac{1}{4}\sqrt{k}$ equals (due to Heron's formula) the area of the weight triangle.

We now formulate some technical results to be exploited later.

Theorem 2. *If the facility W is the solution to the problem (4) for some configuration $\left\{ \frac{P_1}{m_1} \middle| \frac{P_2}{m_2} \middle| \frac{P_3}{m_3} \right\}$ then this facility remains unchanged for the configuration $\left\{ \frac{P_1}{m_1} \middle| \frac{P_2}{m_2} \middle| \tilde{P}_3 \right\}$ with any position of the terminal \tilde{P}_3 in the half-line WP_3 .*

Theorem 3. *The facility W lies in the segment P_3Q_1 . For any position of the terminal P_3 , the facility W lies in the arc of the circle C_1 passing through the points P_1, P_2 and Q_1 . Here*

$$Q_1 = \left(\frac{1}{2}(x_1 + x_2) + \frac{(m_1^2 - m_2^2)(x_1 - x_2) - \sqrt{\mathbf{k}}(y_1 - y_2)}{2m_3^2}, \right. \\ \left. \frac{1}{2}(y_1 + y_2) + \frac{(m_1^2 - m_2^2)(y_1 - y_2) + \sqrt{\mathbf{k}}(x_1 - x_2)}{2m_3^2} \right). \quad (8)$$

Theorem 4. *Let the conditions of Theorem 1 be satisfied. Set*

$$\mathfrak{S}_1 = \begin{vmatrix} 1 & 1 & 1 \\ x & x_2 & x_3 \\ y & y_2 & y_3 \end{vmatrix}, \quad \mathfrak{S}_2 = \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x & x_3 \\ y_1 & y & y_3 \end{vmatrix}, \quad \mathfrak{S}_3 = \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x \\ y_1 & y_2 & y \end{vmatrix}. \quad (9)$$

For any value of the weight m_3 , the optimal facility W lies in the arc of the algebraic curve of the 4th degree given by the equation

$$m_1^2 \mathfrak{S}_2^2 [(x - x_2)^2 + (y - y_2)^2] = m_2^2 \mathfrak{S}_1^2 [(x - x_1)^2 + (y - y_1)^2]. \quad (10)$$

We next treat the four-terminal case.

Assumption 1. *Hereinafter we will treat the case where the terminals $\{P_j\}_{j=1}^4$, while counted counterclockwise, compose a convex quadrilateral $P_1P_2P_3P_4$.*

Stationary points of the function $\sum_{j=1}^4 m_j |WP_j|$ are given by the system of equations

$$\sum_{j=1}^4 \frac{m_j(x - x_j)}{|WP_j|} = 0, \quad \sum_{j=1}^4 \frac{m_j(y - y_j)}{|WP_j|} = 0. \quad (11)$$

Though this system is not an algebraic one with respect to x, y , it can be reduced to this form via successive squaring of every equation. This permits one to apply the procedure of elimination of a variable via computation of the **resultant**. Thereby, the problem of finding the coordinates of the facility W can be reduced to that of resolving a univariate algebraic equation with coefficients polynomially depending on $\{m_j, (x_j, y_j)\}_{j=1}^4$ [9]. The degree of this equation generically equals 12, it is irreducible over \mathbb{Z} , and cannot be expected to be solvable by radicals [1].

3 Bifacility Network for Four Terminals

Assumption 2. *We will assume the weights of the problem (2) to satisfy the restrictions*

$$m < m_1 + m_2, \quad m_1 < m + m_2, \quad m_2 < m + m_1, \quad (12)$$

and

$$m < m_3 + m_4, \quad m_3 < m + m_4, \quad m_4 < m + m_3. \quad (13)$$

From this follows that the values

$$\mathbf{k}_{12} = (m + m_1 + m_2)(m - m_1 + m_2)(m + m_1 - m_2)(-m + m_1 + m_2), \quad (14)$$

$$\mathbf{k}_{34} = (m + m_3 + m_4)(m - m_3 + m_4)(m + m_3 - m_4)(-m + m_3 + m_4) \quad (15)$$

are positive. Additionally we assume the fulfillment of the following inequalities:

$$(m^2 - m_1^2 + m_2^2)/\sqrt{\mathbf{k}_{12}} + (m^2 - m_4^2 + m_3^2)/\sqrt{\mathbf{k}_{34}} > 0, \quad (16)$$

$$(m^2 + m_1^2 - m_2^2)/\sqrt{\mathbf{k}_{12}} + (m^2 + m_4^2 - m_3^2)/\sqrt{\mathbf{k}_{34}} > 0. \quad (17)$$

Theorem 5. Let Assumptions 1 and 2 be fulfilled. Set

$$\begin{aligned} \tau_1 = & \sqrt{\mathbf{k}_{12}} \left[\sqrt{\mathbf{k}_{34}}(x_4 - x_3) - (m^2 + m_3^2 - m_4^2)y_3 - (m^2 - m_3^2 + m_4^2)y_4 \right] \\ & + 2m^2\sqrt{\mathbf{k}_{12}}y_2 + \mathbf{k}_{12}(x_1 - x_2) + (m^2 + m_1^2 - m_2^2) \left[\sqrt{\mathbf{k}_{34}}(y_3 - y_4) \right. \\ & \left. + (m^2 + m_1^2 - m_2^2)x_1 + (m^2 - m_1^2 + m_2^2)x_2 - (m^2 + m_3^2 - m_4^2)x_3 - (m^2 - m_3^2 + m_4^2)x_4 \right], \end{aligned}$$

$$\begin{aligned} \tau_2 = & -\sqrt{\mathbf{k}_{12}} \left[\sqrt{\mathbf{k}_{34}}(x_4 - x_3) - (m^2 + m_3^2 - m_4^2)y_3 - (m^2 - m_3^2 + m_4^2)y_4 \right] \\ & - 2m^2\sqrt{\mathbf{k}_{12}}y_1 - \mathbf{k}_{12}(x_1 - x_2) + (m^2 - m_1^2 + m_2^2) \left[\sqrt{\mathbf{k}_{34}}(y_3 - y_4) \right. \\ & \left. + (m^2 + m_1^2 - m_2^2)x_1 + (m^2 - m_1^2 + m_2^2)x_2 - (m^2 + m_3^2 - m_4^2)x_3 - (m^2 - m_3^2 + m_4^2)x_4 \right], \end{aligned}$$

$$\eta_1 = \frac{1}{\sqrt{\mathbf{k}_{12}}} \left[(m^2 - m_1^2 - m_2^2)\tau_1 - 2m_1^2\tau_2 \right],$$

$$\eta_2 = \frac{1}{\sqrt{\mathbf{k}_{12}}} \left[2m_2^2\tau_1 - (m^2 - m_1^2 - m_2^2)\tau_2 \right]$$

and set the values for $\tau_3, \tau_4, \eta_3, \eta_4$ via the formulae obtained by the cyclic substitution for subscripts

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

in the above expressions for $\tau_1, \tau_2, \eta_1, \eta_2$ correspondingly.

If all the values

$$\delta_1 = \eta_2(x_1 - x_2) + \tau_2(y_2 - y_1), \quad \delta_2 = \eta_1(x_1 - x_2) + \tau_1(y_2 - y_1), \quad (18)$$

$$\delta_3 = \eta_4(x_3 - x_4) + \tau_4(y_4 - y_3), \quad \delta_4 = \eta_3(x_3 - x_4) + \tau_3(y_4 - y_3) \quad (19)$$

and

$$\begin{aligned} \delta = & -\frac{\delta_1(m^2 + m_1^2 - m_2^2)}{\sqrt{\mathbf{k}_{12}}} - \frac{\delta_3(m^2 + m_3^2 - m_4^2)}{\sqrt{\mathbf{k}_{34}}} \\ & + (\eta_1 + \eta_2)(y_1 - y_3) + (\tau_1 + \tau_2)(x_1 - x_3) \end{aligned} \quad (20)$$

are positive then there exists a pair of points W_1 and W_2 lying inside $P_1P_2P_3P_4$ that provides the global minimum value for the function (2). The coordinates of the optimal facility W_1 are as follows:

$$x_* = x_1 - \frac{2\delta_1 m^2 \tau_1}{\sqrt{\mathbf{k}_{12}}\Delta}, \quad (21)$$

$$y_* = y_1 - \frac{2\delta_1 m^2 \eta_1}{\sqrt{\mathbf{k}_{12}}\Delta}, \quad (22)$$

while those of W_2 :

$$x_{**} = x_3 - \frac{2\delta_3 m^2 \tau_3}{\sqrt{\mathbf{k}_{34}}\Delta}, \quad (23)$$

$$y_{**} = y_3 - \frac{2\delta_3 m^2 \eta_3}{\sqrt{\mathbf{k}_{34}}\Delta}. \quad (24)$$

The corresponding minimum value of the function (2) equals

$$\mathfrak{C} = \frac{\sqrt{\Delta}}{4m^3}. \quad (25)$$

Here

$$\Delta = [(\eta_1 + \eta_2)^2 + (\tau_1 + \tau_2)^2]. \quad (26)$$

Proof. (I) We first present some directly verified relations between the values τ -s, η -s and δ -s.

$$\tau_1 = \frac{1}{2m^2} \left[\sqrt{\mathbf{k}_{12}}(\eta_1 + \eta_2) + (m^2 + m_1^2 - m_2^2)(\tau_1 + \tau_2) \right], \quad (27)$$

$$\tau_3 = \frac{1}{2m^2} \left[-\sqrt{\mathbf{k}_{34}}(\eta_1 + \eta_2) - (m^2 + m_3^2 - m_4^2)(\tau_1 + \tau_2) \right], \quad (28)$$

$$\tau_1 + \tau_2 + \tau_3 + \tau_4 = 0, \quad \eta_1 + \eta_2 + \eta_3 + \eta_4 = 0, \quad (29)$$

$$\sum_{j=1}^4 (x_j \tau_j + y_j \eta_j) = \frac{\Delta}{4m^4}; \quad (30)$$

$$\tau_1^2 + \eta_1^2 = \frac{m_1^2}{m^2} \Delta, \quad (31)$$

$$\tau_1 \eta_2 - \tau_2 \eta_1 = \frac{\sqrt{\mathbf{k}_{12}}}{2m^2} \Delta, \quad (32)$$

$$\tau_2 \eta_3 - \tau_3 \eta_2 = \frac{\sqrt{\mathbf{k}_{12}\mathbf{k}_{34}}}{4m^4} \left[\frac{m^2 - m_1^2 + m_2^2}{\sqrt{\mathbf{k}_{12}}} + \frac{m^2 - m_4^2 + m_3^2}{\sqrt{\mathbf{k}_{34}}} \right] \Delta, \quad (33)$$

$$\delta_1 + \delta_3 = (x_1 - x_3)(\eta_1 + \eta_2) - (y_1 - y_3)(\tau_1 + \tau_2), \quad (34)$$

$$2\delta_2 m_2^2 = (m^2 - m_1^2 - m_2^2)\delta_1 - \sqrt{\mathbf{k}_{12}} [(y_1 - y_2)\eta_2 + (x_1 - x_2)\tau_2], \quad (35)$$

$$2\delta_4 m_4^2 = (m^2 - m_3^2 - m_4^2)\delta_3 - \sqrt{\mathbf{k}_{34}} [(y_3 - y_4)\eta_4 + (x_3 - x_4)\tau_4]. \quad (36)$$

(II) Consider the system of equations for determining stationary points of the objective function (2):

$$\frac{\partial F}{\partial x_*} = m_1 \frac{x_* - x_1}{|W_1 P_1|} + m_2 \frac{x_* - x_2}{|W_1 P_2|} + m \frac{x_* - x_{**}}{|W_1 W_2|} = 0, \quad (37)$$

$$\frac{\partial F}{\partial y_*} = m_1 \frac{y_* - y_1}{|W_1 P_1|} + m_2 \frac{y_* - y_2}{|W_1 P_2|} + m \frac{y_* - y_{**}}{|W_1 W_2|} = 0, \quad (38)$$

$$\frac{\partial F}{\partial x_{**}} = m_3 \frac{x_{**} - x_3}{|W_2 P_3|} + m_4 \frac{x_{**} - x_4}{|W_2 P_4|} + m \frac{x_{**} - x_*}{|W_2 W_1|} = 0, \quad (39)$$

$$\frac{\partial F}{\partial y_{**}} = m_3 \frac{y_{**} - y_3}{|W_2 P_3|} + m_4 \frac{y_{**} - y_4}{|W_2 P_4|} + m \frac{y_{**} - y_*}{|W_2 W_1|} = 0. \quad (40)$$

Let us verify the validity of (37). First establish the alternative representations for the coordinates (21) and (22):

$$x_* = x_2 - \frac{2m^2 \delta_2 \tau_2}{\sqrt{\mathbf{k}_{12}} \Delta}, \quad (41)$$

$$y_* = y_2 - \frac{2m^2 \delta_2 \eta_2}{\sqrt{\mathbf{k}_{12}} \Delta}. \quad (42)$$

Indeed, the difference of the right-hand sides of (21) and (41) equals

$$x_1 - x_2 - \frac{2m^2 (\delta_1 \tau_1 - \delta_2 \tau_2)}{\sqrt{\mathbf{k}_{12}} \Delta}$$

and the numerator of the involved fraction can be transformed into

$$\begin{aligned} & \stackrel{(18)}{=} 2m^2 [\tau_1 \eta_2 (x_1 - x_2) + \tau_1 \tau_2 (y_2 - y_1) - \tau_2 \eta_1 (x_1 - x_2) - \tau_2 \tau_1 (y_2 - y_1)] \\ & = 2m^2 (x_1 - x_2) (\tau_1 \eta_2 - \tau_2 \eta_1) \stackrel{(32)}{=} (x_1 - x_2) \sqrt{\mathbf{k}_{12}} \Delta. \end{aligned}$$

The equivalence of (42) and (22) can be demonstrated in a similar manner. Now express the segment lengths:

$$|W_1 P_1| = \sqrt{(x_1 - x_*)^2 + (y_1 - y_*)^2} \stackrel{(21),(22)}{=} \frac{2\delta_1 m^2}{\sqrt{\mathbf{k}_{12}} \Delta} \sqrt{\tau_1^2 + \eta_1^2} \stackrel{(31)}{=} \frac{2m m_1}{\sqrt{\mathbf{k}_{12}}} \frac{\delta_1}{\sqrt{\Delta}} \quad (43)$$

and, similarly,

$$|W_1 P_2| \stackrel{(41),(42)}{=} \frac{2m m_2}{\sqrt{\mathbf{k}_{12}}} \frac{\delta_2}{\sqrt{\Delta}}. \quad (44)$$

With the aid of relations (21), (41), (43) and (44) one can represent the first two terms in the left-hand side of the equality (37) as

$$m_1 \frac{x_* - x_1}{|W_1 P_1|} + m_2 \frac{x_* - x_2}{|W_1 P_2|} = -\frac{m}{\sqrt{\Delta}}(\tau_1 + \tau_2). \quad (45)$$

The third summand in the equality (37) needs more laborious manipulations. We first transform its numerator:

$$x_* - x_{**} \stackrel{(21),(24)}{=} x_1 - x_3 + \frac{2m^2}{\Delta} \left[\frac{\delta_3 \tau_3}{\sqrt{\mathbf{k}_{34}}} - \frac{\delta_1 \tau_1}{\sqrt{\mathbf{k}_{12}}} \right].$$

Now write down the following modification:

$$\begin{aligned} & 2m^2 \left[\frac{\delta_3 \tau_3}{\sqrt{\mathbf{k}_{34}}} - \frac{\delta_1 \tau_1}{\sqrt{\mathbf{k}_{12}}} \right] \stackrel{(27),(28)}{=} \left[-(\eta_1 + \eta_2) - \frac{m^2 + m_3^2 - m_4^2}{\sqrt{\mathbf{k}_{34}}}(\tau_1 + \tau_2) \right] \delta_3 \\ & \quad - \left[(\eta_1 + \eta_2) + \frac{m^2 + m_1^2 - m_2^2}{\sqrt{\mathbf{k}_{12}}}(\tau_1 + \tau_2) \right] \delta_1 \\ & = - \left[(\eta_1 + \eta_2)(\delta_1 + \delta_3) \right. \\ & \quad \left. + (\tau_1 + \tau_2) \left\{ \frac{\delta_1 (m^2 + m_1^2 - m_2^2)}{\sqrt{\mathbf{k}_{12}}} + \frac{\delta_3 (m^2 + m_3^2 - m_4^2)}{\sqrt{\mathbf{k}_{34}}} \right\} \right] \\ & \stackrel{(20)}{=} - \left[(\eta_1 + \eta_2)(\delta_1 + \delta_3) \right. \\ & \quad \left. + (\tau_1 + \tau_2) \{ -\delta + (\eta_1 + \eta_2)(y_1 - y_3) + (\tau_1 + \tau_2)(x_1 - x_3) \} \right] \\ & = \delta(\tau_1 + \tau_2) - (\eta_1 + \eta_2) \left[\delta_1 + \delta_3 + (\tau_1 + \tau_2)(y_1 - y_3) - (\eta_1 + \eta_2)(x_1 - x_3) \right] - \\ & \quad \Delta(x_1 - x_3) \stackrel{(34)}{=} \delta(\tau_1 + \tau_2) - \Delta(x_1 - x_3). \end{aligned}$$

Finally,

$$x_* - x_{**} = x_1 - x_3 + \frac{\delta(\tau_1 + \tau_2) - \Delta(x_1 - x_3)}{\Delta} = \frac{\delta(\tau_1 + \tau_2)}{\Delta}.$$

Similarly the following equality can be deduced: $y_* - y_{**} = \frac{\delta(\eta_1 + \eta_2)}{\Delta}$, and both formulae yield

$$|W_1 W_2| = \sqrt{(x_* - x_{**})^2 + (y_* - y_{**})^2} = \frac{\delta}{\sqrt{\Delta}}. \quad (46)$$

Therefore, the last summand of equality (37) takes the form

$$mz \frac{x_* - x_{**}}{|W_1 W_2|} = m \frac{\delta(\tau_1 + \tau_2)\sqrt{\Delta}}{\delta\Delta} = m \frac{\tau_1 + \tau_2}{\sqrt{\Delta}}.$$

Summation this with (45) yields 0 and this completes the proof of (37).

The validity of the remaining equalities (38)–(40) can be established in a similar way.

(III) We now deduce the formula (25) for the network cost. With the aid of the formulae (43), (44), (46) and their counterparts for the segment lengths $|W_2 P_3|$ and $|W_2 P_4|$, one gets

$$\begin{aligned} & m_1|W_1 P_1| + m_2|W_1 P_2| + m_3|W_2 P_3| + m_4|W_2 P_4| + m|W_1 W_2| \\ &= \frac{2m}{\sqrt{\Delta}} \left(\frac{m_1^2 \delta_1}{\sqrt{\mathbf{k}_{12}}} + \frac{m_2^2 \delta_2}{\sqrt{\mathbf{k}_{12}}} + \frac{m_3^2 \delta_3}{\sqrt{\mathbf{k}_{34}}} + \frac{m_4^2 \delta_4}{\sqrt{\mathbf{k}_{34}}} + \frac{\delta}{2} \right) \\ &\stackrel{(20)}{=} \frac{2m}{\sqrt{\Delta}} \left\{ \frac{\delta_1}{2\sqrt{\mathbf{k}_{12}}}(-m^2 + m_2^2 + m_1^2) + \frac{\delta_3}{2\sqrt{\mathbf{k}_{34}}}(-m^2 + m_3^2 + m_4^2) + \frac{m_2^2 \delta_2}{\sqrt{\mathbf{k}_{12}}} + \frac{m_4^2 \delta_4}{\sqrt{\mathbf{k}_{34}}} \right. \\ &\quad \left. + \frac{1}{2}(\eta_1 + \eta_2)(y_1 - y_3) + \frac{1}{2}(\tau_1 + \tau_2)(x_1 - x_3) \right\} \\ &\stackrel{(35),(36)}{=} \frac{2m}{\sqrt{\Delta}} \left\{ -\frac{1}{2}(y_1 - y_2)\eta_2 - \frac{1}{2}(x_1 - x_2)\tau_2 - \frac{1}{2}(y_3 - y_4)\eta_4 - \frac{1}{2}(x_3 - x_4)\tau_4 \right. \\ &\quad \left. + \frac{1}{2}(\eta_1 + \eta_2)(y_1 - y_3) + \frac{1}{2}(\tau_1 + \tau_2)(x_1 - x_3) \right\} \\ &= \frac{m}{\sqrt{\Delta}} \{ y_1\eta_1 + y_2\eta_2 + y_4\eta_4 + x_1\tau_1 + x_2\tau_2 + x_4\tau_4 - x_3(\tau_1 + \tau_2 + \tau_4) \\ &\quad - y_3(\eta_1 + \eta_2 + \eta_4) \} \\ &\stackrel{(29)}{=} \frac{m}{\sqrt{\Delta}} \sum_{j=1}^4 (x_j\tau_j + y_j\eta_j) \stackrel{(30)}{=} \frac{\sqrt{\Delta}}{4m^3}. \end{aligned}$$

For the proof of the two last statements we refer to [7].

(IV) The facilities W_1 and W_2 providing the solution to the problem (2) lie inside the quadrilateral $P_1 P_2 P_3 P_4$.

(V) The function (2) is strictly convex inside the *convex* (due to Assumption 1) domain given as the Cartesian product $P_1 P_2 P_3 P_4 \times P_1 P_2 P_3 P_4$. Therefore the solution of the system (37)–(40) provides the global minimum value for this function. \square

The result of Theorem 5 claims that the bifacility Weber problem for four terminals is solvable by radicals, and thus we get a natural extension of the three-terminal problem solution given in Theorem 1. An additional correlation between these two results can be watched, namely that the denominators of all the formulae for the facilities coordinates contain the explicit expression for the cost of the corresponding network. It looks like every facility “knows” the cost of the network that includes this point.

4 Solution Analysis

Analytical solution obtained in the previous section provides one with an opportunity to analyze the dynamics of the network under variation of the parameters of the configuration and to find the bifurcation values for these parameters, i.e. those responsible for the topology degeneracy. We first treat the cases where either the coordinates of a terminal or the corresponding weight are varied.

Theorem 6. *For any position of the terminal P_3 , the facility W_1 lies in the arc of the circle C_1 passing through the points P_1, P_2 and $Q_1 = (q_{1x}, q_{1y})$ given by the formula (8) where substitution $m_3 \rightarrow m$ is made. At the same time, the facility W_2 lies in the arc of the circle C_3 passing through the points Q_1, P_4 and \tilde{Q}_3 . Here \tilde{Q}_3 is given by (8) where substitution*

$$\begin{pmatrix} x_1, y_1 \\ x_4, y_4 \end{pmatrix} \bigg| \begin{pmatrix} x_2, y_2 \\ q_{1x}, q_{1y} \end{pmatrix} \bigg| \begin{pmatrix} m_1 \\ m_4 \end{pmatrix} \bigg| \begin{pmatrix} m_2 \\ m \end{pmatrix} \bigg| m_3$$

is applied to.

Example 1. *For the configuration*

$$\left\{ \begin{array}{c} P_1 = (1, 5) \\ m_1 = 3 \end{array} \bigg| \begin{array}{c} P_2 = (2, 1) \\ m_2 = 2 \end{array} \bigg| \begin{array}{c} P_3 \\ m_3 = 3 \end{array} \bigg| \begin{array}{c} P_4 = (6, 7) \\ m_4 = 4 \end{array} \bigg| m = 4 \right\},$$

find the loci of the facilities W_1 and W_2 under variation of the terminal P_3 moving somehow from the starting position at $(9, 2)$ towards P_2 .

Solution. The trajectory of P_3 does not influence those of W_1 and W_2 , i.e. both facilities do not leave the corresponding arcs for any drift of P_3 until the latter swashes the line $\mathcal{L} = \tilde{Q}_3 W$ (Fig. 1). At this moment, W_1 collides with W_2 in the point

$$W = \left(\frac{867494143740435 + 114770004066285\sqrt{33} + 14973708000030\sqrt{55} + 19296850969306\sqrt{15}}{435004929875940}, \frac{581098602680450 + 10154769229801\sqrt{15} + 9689425113917\sqrt{55} - 18326585102850\sqrt{33}}{145001643291980} \right),$$

$\approx (3.936925, 4.048287)$ which stands for the second point of intersection of the circles C_1 and C_3 , and yields a solution to the unifacility Weber problem (1) for the configuration $\left\{ \frac{P_1}{m_1} \big| \frac{P_2}{m_2} \big| \frac{P_3}{m_3} \big| \frac{P_4}{m_4} \right\}$. When P_3 crosses the line \mathcal{L} , the solution to the bifacility Weber problem (2) does not exist (while the unifacility counterpart (1) still possesses a solution). \square

$$m^2 \begin{vmatrix} 1 & 1 & 1 \\ x & q_{1x} & x_3 \\ y & q_{1y} & y_3 \end{vmatrix}^2 [(x - x_4)^2 + (y - y_4)^2] = m_4^2 \begin{vmatrix} 1 & 1 & 1 \\ x & x_3 & x_4 \\ y & y_3 & y_4 \end{vmatrix}^2 [(x - q_{1x})^2 + (y - q_{1y})^2]. \quad (47)$$

We finally treat the case of the variation of the parameter directly responsible for the inter-facilities connection; the bifurcation equation is now determined by (20).

Example 2. *For the configuration*

$$\left\{ \begin{array}{c} P_1 = (1, 5) \\ m_1 = 3 \end{array} \middle| \begin{array}{c} P_2 = (2, 1) \\ m_2 = 2 \end{array} \middle| \begin{array}{c} P_3 = (7, 2) \\ m_3 = 3 \end{array} \middle| \begin{array}{c} P_4 = (6, 7) \\ m_4 = 4 \end{array} \middle| m \right\},$$

find the loci of the facilities W_1 and W_2 under variation of the weight m within the interval $[2, 4.8]$.

Solution. Due to (46), the trajectories of W_1 and W_2 meet when m coincides with a zero of the equation $\delta(m) = 0$. The latter can be reduced to an algebraic one $24505 m^{20} - 3675750 m^{18} + \dots + 25596924755077 = 0$ with a (closest to $m = 4$) zero $m_{0,1} \approx 4.326092$. The collision point W yields the solution to the unifacility Weber problem (1) for the configuration $\left\{ \begin{array}{c} P_1 \\ m_1 \end{array} \middle| \begin{array}{c} P_2 \\ m_2 \end{array} \middle| \begin{array}{c} P_3 \\ m_3 \end{array} \middle| \begin{array}{c} P_4 \\ m_4 \end{array} \right\}$. Its coordinates $(x_*, y_*) \approx (4.537574, 4.565962)$ satisfy the 10th degree algebraic equations over \mathbb{Z} , and this time (as opposed to the variant from Example 1) one cannot expect them to be expressed by radicals.

This scenario demonstrates a paradoxical phenomenon: the weight m increase forces the facilities to a collision, i.e. to a network configuration where its influence disappears completely.

When m decreases from $m = 4$, the facility W_1 moves towards P_1 while W_2 moves towards P_4 . The first drift is faster than the second one: W_1 approaches P_1 when m coincides with a zero of the equation $\delta_1(m) = 0$. The latter can be reduced to an algebraic one with a zero $m_{0,2} \approx 3.145546$. \square

Theorem 9. *If the optimal bifacility network exists, it is less costly than the unifacility one.*

Proof. If the cost (25) is considered as the function of the configuration parameters then the following identities are valid:

$$\frac{\partial \mathfrak{C}^2}{\partial m_1} \equiv \frac{m_1 \delta_1}{m^2 \sqrt{\mathbf{k}_{12}}}, \quad \frac{\partial \mathfrak{C}^2}{\partial m_2} \equiv \frac{m_2 \delta_2}{m^2 \sqrt{\mathbf{k}_{12}}}, \quad \frac{\partial \mathfrak{C}^2}{\partial m_3} \equiv \frac{m_3 \delta_3}{m^2 \sqrt{\mathbf{k}_{34}}}, \quad \frac{\partial \mathfrak{C}^2}{\partial m_4} \equiv \frac{m_4 \delta_4}{m^2 \sqrt{\mathbf{k}_{34}}}$$

$$\text{and} \quad \frac{\partial \mathfrak{C}^2}{\partial m} \equiv \frac{\delta}{2m^3}.$$

The last one results in $\partial \mathfrak{C} / \partial m = \delta / (4m^3 \mathfrak{C})$. Therefore for any specialization of the weights $\{m_j\}_{j=1}^4$, the function $\mathfrak{C}(m)$ increases to its maximal value at the positive zero of $\delta(m)$. \square

5 Five Terminals

In order to extend an analytical approach developed in Sect. 3 to the multifacility Weber problem, we first demonstrate here an alternative approach for solution of the four-terminal problem (2). It is based on the reduction of this problem to the pair of the three-terminal Weber problems. We will utilize abbreviations $\{4t2f\}$ and $\{3t1f\}$ for the corresponding problems.

Assume that solution for the $\{4t2f\}$ -Weber problem (2) exists. Then the system of equations (37)–(40) providing the coordinates of the facilities could be split into two subsystems. Comparing equations (37) and (38) with (6) and (7) permits one to claim that the optimal facility W_1 coincides with its counterpart for the $\{3t1f\}$ -Weber problem for the configuration $\{P_1|P_2|W_2\}$. A similar statement is also valid for the facility W_2 , i.e. it is the solution to the Weber problem for the configuration $\{P_3|P_4|W_1\}$. From this point of view, it looks like the four-terminal Weber problem can be reduced to the pair of the three-terminal ones. However, this reduction should be modified since the loci of the facilities W_2 or W_1 remain still undetermined. The result of Theorems 2 and 3 permits one to replace these facilities by those with known positions.

Theorem 10. *If the solution to the $\{4t2f\}$ -Weber problem (2) exists then the facility W_2 coincides with the solution to the $\{3t1f\}$ -Weber problem for the configuration $\{P_3|P_4|Q_1\}$. Here Q_1 is the point defined by (8) with the substitution $m_3 \rightarrow m$. A similar statement is valid for the terminal W_1 : it coincides with the solution to the $\{3t1f\}$ -Weber problem for the configuration $\{P_1|P_2|Q_2\}$ where the coordinates for Q_2 are obtained via (8) where the substitution for the indices $1 \rightarrow 3, 2 \rightarrow 4$ is made together with $m_3 \rightarrow m$.*

This theorem claims that the four-terminal Weber problem can be solved by its reduction to the three-terminal counterpart via a formal replacement of a pair of the *real* terminals, say P_3 and P_4 , by a single *phantom* terminal Q_2 . This reduction algorithm is similar to that used for construction of the Steiner minimal tree (firstly introduced by Gergonne as early as in 1810, and 150 years later rediscovered by Melzak [2]). The approach can be evidently extended to the general case of $n \geq 5$ terminals as will be clarified by the following example.

Example 3. *Find the coordinates of the facilities W_1, W_2, W_3 that minimize the cost*

$$m_1|P_1W_1| + m_2|P_2W_1| + m_3|P_3W_2| + m_4|P_4W_2| + m_5|P_5W_3| + \tilde{m}_{1,3}|W_1W_3| + \tilde{m}_{2,3}|W_2W_3| \quad (48)$$

for the following configuration:

$$\left\{ \begin{array}{l} P_1 = (1, 6) \mid P_2 = (5, 1) \mid P_3 = (11, 1) \mid P_4 = (15, 3) \mid P_5 = (7, 11) \mid \tilde{m}_{1,3} = 10 \\ m_1 = 10 \mid m_2 = 9 \mid m_3 = 8 \mid m_4 = 7 \mid m_5 = 13 \mid \tilde{m}_{2,3} = 12 \end{array} \right\}.$$

Solution. (I) To reduce the problem to the $\{4\mathbf{t}2\mathbf{f}\}$ -case, replace a pair of the terminals P_1 and P_2 by the point Q_1 defined by the formula (8) where the substitution $m_3 \rightarrow \tilde{m}_{1,3}$ is made.

$$Q_1 = \left(-\frac{9}{40}\sqrt{319} + \frac{131}{50}, -\frac{9}{50}\sqrt{319} + \frac{159}{40} \right) \approx (-1.398628, 0.760097).$$

(II) Solve the $\{4\mathbf{t}2\mathbf{f}\}$ -problem for the configuration $\left\{ \begin{smallmatrix} P_5 \\ m_5 \end{smallmatrix} \middle| \begin{smallmatrix} Q_1 \\ \tilde{m}_{1,3} \end{smallmatrix} \middle| \begin{smallmatrix} P_3 \\ m_3 \end{smallmatrix} \middle| \begin{smallmatrix} P_4 \\ m_4 \end{smallmatrix} \middle| \begin{smallmatrix} \tilde{m}_{2,3} \end{smallmatrix} \right\}$ via formulae (21)–(24) and obtain the coordinates for the facilities

$$W_2 \approx (10.441211, 3.084533) \text{ and } W_3 \approx (7.191843, 5.899268).$$

(III) Return P_1 and P_2 instead of Q_1 and solve the $\{3\mathbf{t}1\mathbf{f}\}$ -Weber problem for the configuration $\left\{ \begin{smallmatrix} P_1 \\ m_1 \end{smallmatrix} \middle| \begin{smallmatrix} P_2 \\ m_2 \end{smallmatrix} \middle| \begin{smallmatrix} W_3 \\ \tilde{m}_{1,3} \end{smallmatrix} \right\}$ by the formulae of Theorem 1: $W_1 \approx (4.750727, 4.438893)$ (Fig. 2). We emphasize, that the coordinates of the facilities can be expressed by radicals similar to the following expression for the minimal cost of the network

$$\begin{aligned} \mathfrak{C} = & \frac{\sqrt{10}}{80} (4158\sqrt{87087} + 773402\sqrt{231} + 271890\sqrt{319} + 247470\sqrt{143} \\ & + 326403\sqrt{609} + 104181\sqrt{273} - 4455\sqrt{377} + 15216515)^{1/2} \approx 267.229644. \end{aligned}$$

□

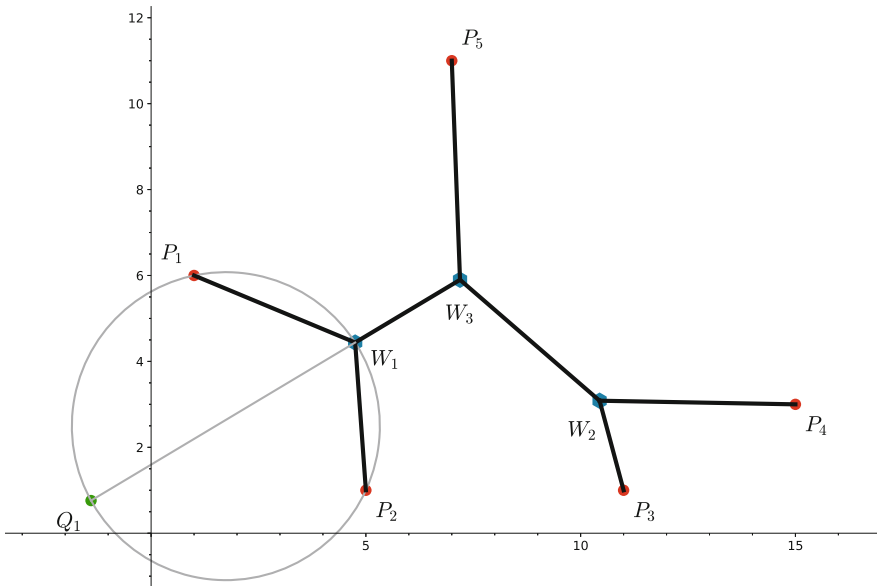


Fig. 2. Example 3. Weber network construction for five terminals

The reduction procedure illuminated in the previous example, in the general case should be accompanied by the conditions similar to those from Theorem 5.

We conclude this section with formulation of two problems for further research. The first one, for simplicity, is given in terms of the last example:

Find the pair of the weights $(\tilde{m}_{1,3}, \tilde{m}_{2,3})$ with the minimal possible sum $\tilde{m}_{1,3} + \tilde{m}_{2,3}$ such that the corresponding optimal network contains a single facility.

The second problem consists in proving (or disproving) of the following

Conjecture. The $\{n \text{ terminals } \ell \text{ facilities}\}$ -Weber problem (3) is solvable by radicals if $\ell = n - 2$ and the valency of every facility in the network equals 3.

6 Conclusions

We provide an analytical solution to the bifacility Weber problem (2) approving thereby the geometric solution by G. Pick. We also formulate the conditions for the existence of the network in a prescribed topology and construct the solution for five terminals.

Several problems for further investigations are mentioned in Section 5. One extra problem concerns the treatment of distance depending functions like $F_L(P) = \sum_{j=1}^n m_j |PP_j|^L$ with different exponents $L \in \mathbb{Q} \setminus 0$. The choice $L = -1$ corresponds to Newton or Coulomb potential. It turns out that the stationary point sets of all the functions $\{F_L\}$ can be treated in the universal manner [9]. We hope to discuss these issues in forthcoming papers.

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