On Stationary Points of Distance Depending Potentials

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Abstract. We continue investigations started in the previous publications by the authors (*LNCS*, volumes 8136 (2013) and 9570 (2016)). The structure of stationary point sets is established for the family of functions given as linear combinations of an exponent L of Euclidean distances from a variable point to the fixed points in 2D and 3D spaces. We compare the structure of the stationary point sets for several values of the exponent L, focusing ourselves mainly onto the cases of Coulomb potential and Weber facility location problem. We develop the analytical approach to the problem aiming at finding the exact number of stationary points and their location in relation to the parameters involved.

Keywords: Stationary points · Coulomb potential · Weber problem

1 Introduction

Given the coordinates of $K \geq 3$ points $\{P_j\}_{j=1}^K \subset \mathbb{R}^n$, the problem is to find the exact number and the coordinates of the stationary points for the function

$$F(P) = \sum_{j=1}^{K} m_j \left| PP_j \right|^L , \quad P = (x_1, \dots, x_n) \in \mathbb{R}^n \ . \tag{1}$$

Here $\{m_j\}_{j=1}^K$ are assumed to be real non-negative numbers, the exponent $L \in \mathbb{R}$ is nonzero while $|\cdot|$ stands for the Euclidean distance.

For $L = 2$, the problem has a well-known solution: the center of mass (barycenter) $P_* = \sum_{j=1}^K m_j P_j / \sum_{j=1}^K m_j$ provides the global minimum for the function $F(P)$. For $L = 1$, the problem is know as the generalized Fermat–Torricelli or the Weber problem, and it is the origin of the branch of Operation Research known as Facility Location. For this case, a unique stationary point might exist for (1), and for the specialization $n = 2, K = 3$, its coordinates can be expresses by radicals via the parameters of the function [1].

For $L = -1$ and $n = 3$, the problem can be viewed to as a classical electrostatics one with the function (1) representing the Coulomb potential of the charges $\{m_j\}_{j=1}^K$ placed at fixed positions $\{P_j\}_{j=1}^K$ in the space. In despite of its classical looking formulation, the problem has not been given a systematic exploration — with the exception of some special configurations [2]. The difficulty of the problem can be acknowledged also from the state of the art with its part known as

Maxwell's Conjecture [3]. The total number of stationary points of any configuration with K charges in $\mathbb{R}^3$ never exceeds $(K-1)^2$.

After more than a century and a half from its formulation, this conjecture has recently attracted attention in [4]. It remains still open even for the case of $K=3$ charges [5].

Aside from the stated problem of localization of stationary points corresponding to minimum (or *stable* equilibria for the potential), we also treat the *parameter synthesis problem.* Namely, we look for the largest possible domain $\mathbb{P}$ in the parameter space $\mathbb{R}^K$ such that for any specialization of parameter vector $(m_1,\dots,m_K)$ from this domain, there exists at least one local minimum for (1). On the other hand, we look for the set $\mathbb{S}$ in coordinate space $\mathbb{R}^n$ where every point might be made a stationary minimum one for (1) by a suitable specialization of $(m_1,\dots,m_K)\in\mathbb{P}$. Every set $\mathbb{S}$ or $\mathbb{P}$ will be hereinafter referred to as the **stability domain** in the corresponding space.

Although the stated problem hardly expect the closed form analytical solution, the latter can be suggested for sufficiently wide class of functions (1). To illuminate this statement, we tackle the case of function (1) where $K=n+1$.

2 Multidimensional case

Stationary points of the function

$$F(P)=\sum_{j=1}^{n+1} m_j |PP_j|^L \tag{2}$$

are given by the system

$$\partial F/\partial x_1=0,\dots,\partial F/\partial x_n=0\,. \tag{3}$$

These equations are linear homogeneous ones with respect to $\{m_j\}_{j=1}^{n+1}$. Their elimination from this system leads to the following

Theorem 1. *Let the points $\{P_j=(x_{j1},\dots,x_{jn})\}_{j=1}^{n+1}$ be chosen such that the condition*

$$V=\left|\begin{array}{llll} 1 & 1 & \dots & 1 \\ x_{11} & x_{21} & \dots & x_{n+1,1} \\ \vdots & \vdots & & \vdots \\ x_{1n} & x_{2n} & \dots & x_{n+1,n} \end{array}\right| > 0 \tag{4}$$

is satisfied. Denote by V_j the determinant obtained on replacing the jth column of (4) by the column[1] $[1,x_1,\dots,x_n]^{\top}$. *Any solution to the system (3) is a solution*

[1] Hereinafter $^{\top}$ denotes transposition.

to the system

$$m_1 : m_2 : \cdots : m_{n+1} = |PP_1|^{2-L} V_1 : |PP_2|^{2-L} V_2 : \cdots : |PP_{n+1}|^{2-L} V_{n+1} \, . \quad (5)$$

Since a solution to system (3) is invariant under substitution $\{m_j \to c m_j\}_{j=1}^{n+1}$, $c \neq 0$, it is possible to assume that at this solution the following relations

$$\left\{ m_j = |PP_j|^{2-L} V_j \right\}_{j=1}^{n+1} \quad (6)$$

are valid up to a common numerical factor. This permits one to find the boundary for stability domain $\mathbb{S}$. Indeed, this boundary is defined by the condition for the Hessian $\mathcal{H}(P)$ of (2) to lose the property of positive definiteness, and this happens when

$$\det \mathcal{H}(P) = 0 \quad (7)$$

at some stationary point P_* of $F(P)$. Joint fulfillment of the equalities (3) and (7) at this point for some specialization of parameters means then that it is a multiple one for the system (3). Namely, its appearance is due to collision of (generically) two non-degenerate stationary points of the function $F(P)$ when the parameter vector $(m_1, \ldots, m_{n+1})$ tends to a bifurcation point lying at the boundary of the stability domain $\mathbb{P}$ in the parameter space.

Since at this point the relationships (6) are valid, one can eliminate the parameters $\{m_j\}_{j=1}^{n+1}$ from (7). This results in an equation for the manifold in $\mathbb{R}^n$ yielding the boundary of the stability domain $\mathbb{S}$ provided the latter is not empty. We detail the structure of this manifold for $n \in \{2, 3\}$ in the foregoing sections, and restrict ourselves here with the following condition for the emptiness of $\mathbb{S}$.

Theorem 2. *For $2 - n \leq L < 0$, none of the stationary point of (2) is a point of minimum.*

For $L > 0$ or $L < 2 - n$, the principal existence of the point of minimum is confirmed by the following

Example 1. Let the points $\{P_j\}_{j=1}^{n+1}$ compose a regular n-dimensional simplex in $\mathbb{R}^n$ centered at $P_* = \mathbb{O}$. For instance, for $n = 3$, one may take

$$P_1 = \begin{pmatrix} -1/3 \\ -\sqrt{2}/3 \\ -\sqrt{2/3} \end{pmatrix}, P_2 = \begin{pmatrix} -1/3 \\ \sqrt{8}/3 \\ 0 \end{pmatrix}, \; P_3 = \begin{pmatrix} -1/3 \\ -\sqrt{2}/3 \\ \sqrt{2/3} \end{pmatrix}, P_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} . \quad (8)$$

For the function $F(P) = \sum_{j=1}^{n+1} |PP_j|^L$, the point $P_* = \mathbb{O} \in \mathbb{R}^n$ is evidently a stationary one. Hessian $\mathcal{H}(\mathbb{O})$ possesses a single eigenvalue (of the multiplicity n) equal (up to a positive factor) to $L(L + n - 2)$. Therefore, $\mathbb{O}$ is the point of minimum iff $L > 0$ or $L < 2 - n$. $\square$

As for the stability domain $\mathbb{P}$ in the parameter space, its boundary can be obtained on elimination of the variables $x_1, \ldots, x_n$ from equation (7) using system (3). Compared with the previously considered procedure of elimination of parameters, this time one cannot expect even the algebraicity of the procedure. Only for the case of the rationality of the exponent L, the system (3) can be reduced to an algebraic one. Elimination of variables from such a system can be organized with the aid of the multivariate resultant computation or via the Gröbner basis construction [6].

3 The 2D case

The planar counterpart of (5) is as follows

$$m_1 : m_2 : m_3 = |PP_1|^{2-L} S_1 : |PP_2|^{2-L} S_2 : |PP_3|^{2-L} S_3 \, . \tag{9}$$

Here $\{P_j = (x_j, y_j)\}_{j=1}^3, P = (x, y)$ and

$$S_1(x,y) = \begin{vmatrix} 1 & 1 & 1 \\ x & x_2 & x_3 \\ y & y_2 & y_3 \end{vmatrix}, \ S_2(x,y) = \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x & x_3 \\ y_1 & y & y_3 \end{vmatrix}, \ S_3(x,y) = \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x \\ y_1 & y_2 & y \end{vmatrix} .$$

Theorem 3. *Let the points P_1, P_2, P_3 be noncollinear and be counted counterclockwise. If $L \ge 1$ then the stability domain $\mathbb{S}$ in the coordinate plane coincides with the interior of the triangle $P_1P_2P_3$, i.e. any point $P_* = (x_*, y_*)$ inside the triangle is the point of minimum for the function*

$$F_*(P) = \sum_{j=1}^{3} m_j^* \, |PP_j|^L \quad \textit{where} \quad \{m_j^* = |P_*P_j|^{2-L} S_j(x_*, y_*)\}_{j=1}^3 \, . \tag{10}$$

If $L < 1, L \ne 0$ then the boundary of the stability domain $\mathbb{S}$ is given by the equation

$$\widetilde{\Phi}_L(x,y) := \frac{S_1(x,y)S_2(x,y)S_3(x,y)}{|PP_1|^2|PP_2|^2|PP_3|^2} \sum_{j=1}^{3} S_j(x,y)|PP_j|^2 + \frac{L-1}{(L-2)^2} S^2 = 0 \tag{11}$$

where

$$S = S_1 + S_2 + S_3 \equiv \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} .$$

Proof. The characteristic polynomial of the Hessian at any stationary point P_* of the function $F(P)$ can be represented as follows [1]:

$$\det(\lambda I - \mathcal{H}(P_*)) = \lambda^2 - L^2 S \lambda + L^2 (L-2)^2 \widetilde{\Phi}_L(x_*, y_*) \, . \tag{12}$$

It turns out that for $L < 1, L \ne 0$, the inequality $\widetilde{\Phi}_L(x,y) > 0$ provides a nonempty domain inside the triangle $P_1P_2P_3$, and if the point P_* lies within this domain then all the zeros of (12) are positive and therefore P_* is point of minimum for (10). □

As for the localization of stability domain in the parameter space, this problem is rather more difficult since the variables x and y to be eliminated from (9) are involved in it in a highly nonlinear manner.

Example 2. Let $P_1 = (1,1), P_2 = (5,1), P_3 = (2,6)$. For the Coulomb potential $F(P) = 1/|PP_1| + m_2/|PP_2| + m_3/|PP_3|$, stability domain $\mathbb{P}$ in the (m_2, m_3)-plane has a boundary which is obtained via resultant computation for the equations

$$m_2^2 S_1^2 |PP_1|^6 - S_2^2 |PP_2|^6 = 0,\ m_2^2 S_3^2 |PP_3|^6 - m_3^2 S_2^2 |PP_2|^6 = 0$$

accomplished with the condition (7); this time the variables x and y are to be eliminated. This results in the algebraic equation $\Psi(m_2, m_3) = 0$ where $\Psi(m_2, m_3) \in \mathbb{Z}[m_2, m_3], \deg \Psi = 48$ and the coefficients of the magnitude of up to 10^{80}. The details of computation and the image of both domains $\mathbb{S}$ and $\mathbb{P}$ are presented in [5]. □

Representation (9) permits one to trace the curve of stationary points under variation of an extra parameter of the considered function, namely the exponent L.

Theorem 4. *For any specialization of $\{m_j, P_j\}_{j=1}^3$, the stationary points of the function (10) lie in the curve*[2]

$$(\log|PP_2| - \log|PP_3|)\log\frac{S_1}{m_1} + (\log|PP_3| - \log|PP_1|)\log\frac{S_2}{m_2}$$
$$+ (\log|PP_1| - \log|PP_2|)\log\frac{S_3}{m_3} = 0 \qquad (13)$$

Proof. From (9) it follows that

$$\begin{cases} \log S_2/m_2 - \log S_1/m_1 - (2-L)(\log|PP_1| - \log|PP_2|) = 0, \\ \log S_3/m_3 - \log S_1/m_1 - (2-L)(\log|PP_1| - \log|PP_3|) = 0. \end{cases} \qquad (14)$$

Elimination of L results in (13). □

Though the curve (13) is not an algebraic one, its depiction does not cause trouble.

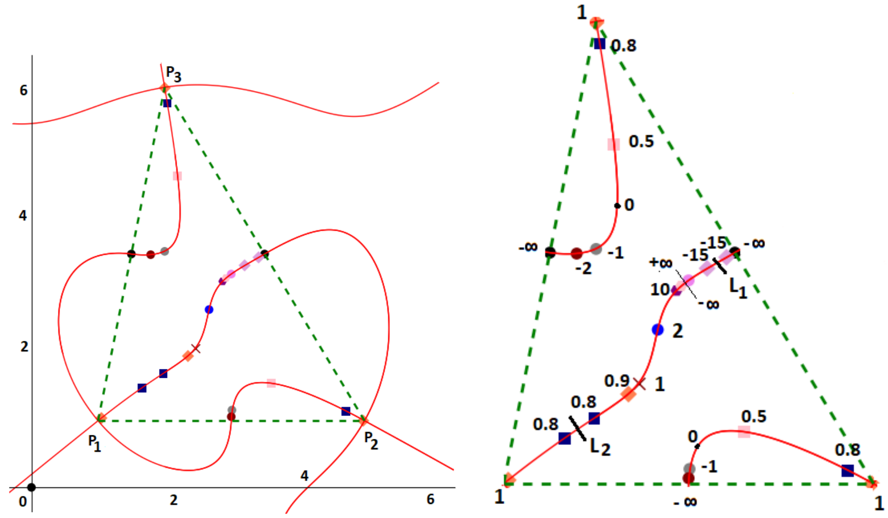
Example 3. For $m_1 = m_2 = m_3 = 1$ and $P_1 = (1,1), P_2 = (5,1), P_3 = (2,6)$, the curve (13) is plotted in Fig. 1 (a).

It passes through nearly all the significant points of the triangle $P_1P_2P_3$, namely its vertices ($L \to 1$), the midpoints of the sides ($L \to -\infty$), centroid ($L = 2$), circumcenter ($L \to +\infty$ and $L \to -\infty$), Fermat–Torricelli point ($L = 1$). The two extra points in the curve

$$\left(\frac{8}{3} \pm \frac{\sqrt{-6+\sqrt{61}}}{3}, \frac{8}{3} \mp \frac{\sqrt{6+\sqrt{61}}}{3}\right) \approx \{(3.1151,\ 1.4279); (2.2181, 3.9054)\}$$

[2] With the logarithm considered to an arbitrary positive base.

corresponding to $L \to 0$ are the stationary points of the logarithmic potential $\log|PP_1| + \log|PP_2| + \log|PP_3|$ (or, equivalently, for $|PP_1| \cdot |PP_2| \cdot |PP_3|$).



(a) (b)

Fig. 1. Example 3. Stationary points for different values of the exponent L.

The bifurcation values for the exponent L are obtained from the condition that equations (14) have a multiple zero with respect to x and y. This is equivalent to vanishment of their Jacobian. It turns out that the latter equals $(L-2)^2\widetilde{\Phi}_L(x,y)/(S_1S_2S_3)$ with $\widetilde{\Phi}_L(x,y)$ defined by (11). Resolving the obtained non-algebraic system yields two bifurcation values for L, namely $L_1 \approx -13.5023$ and $L_2 \approx 0.7948$ (Fig. 1 (b)). An extra bifurcation value is $L_3 = 1$. When L approaches L_1 from the left the two stationary points tend to collision point at $\approx (3.3099, 3.3907)$; when L approaches L_2 from the right the two stationary points tend to collision at $\approx (1.8354, 1.6141)$. When L approaches 1 from the left, the three stationary points tend to P_1, P_2 and P_3. Bifurcation values L_1, L_2, L_3 separate the intervals in the L-axis corresponding to distinct numbers of stationary points for the function (10). The latter possesses four stationary points if $L < L_1$ or $L_2 < L < 1$, two points if $L_1 < L < L_2$, and a single point if $L \geq 1$. $\square$

4 The 3D case

We next treat the case $n = 3$, i.e.,

$$F(P) = \sum_{j=1}^{4} m_j \left|PP_j\right|^L, \ P = (x, y, z), \ \{P_j = (x_j, y_j, z_j)\}_{j=1}^4 \subset \mathbb{R}^3 \qquad (15)$$

In the following result we give a corrected version of one erroneous statement from [1].

Theorem 5. *Let the points* $\{P_j\}_{j=1}^4$ *satisfy the assumptions of Theorem 1. If* $L \geq 2$ *then the stability domain* $\mathbb{S}$ *for the function (15) in the coordinate space coincides with the interior of the simplex* $P_1P_2P_3P_4$. *If* $L \in [-1;0]$ *then the domain is empty (v. Theorem 2). Else the boundary for the domain of stability is given by the equation*

$$\widetilde{\Phi}_L(x,y,z) = (L-1)V^3 + (L-2)^2 V t_2(x,y,z) + (L-2)^3 t_3(x,y,z) = 0 \quad (16)$$

where

$$t_2(x,y,z) = \sum_{1\le j<k\le 4} V_j V_k \frac{\det(M_{jk}\cdot M_{jk}^{\top})}{|P_*P_j|^2|PP_k|^2}, \ M_{jk} = \begin{bmatrix} x-x_j & y-y_j & z-z_j \\ x-x_k & y-y_k & z-z_k \end{bmatrix};$$

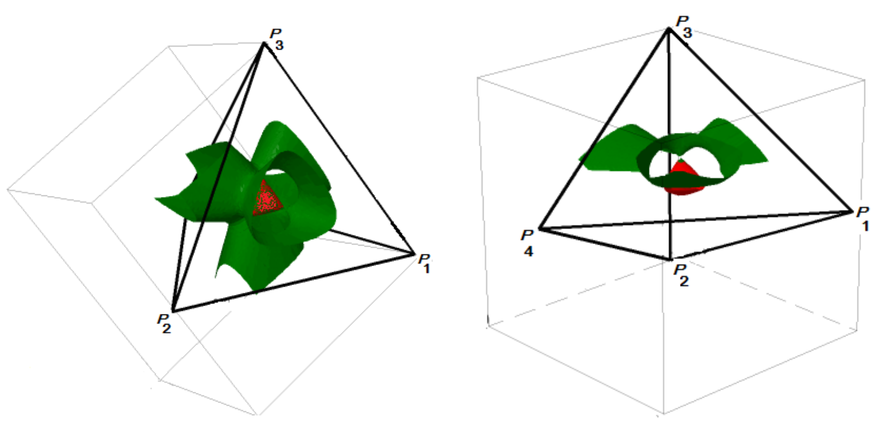
and

$$t_3(x,y,z) = \frac{V_1V_2V_3V_4}{|PP_1|^2|PP_2|^2|PP_3|^2|PP_4|^2} \sum_{j=1}^{4} V_j \left|PP_j\right|^2 .$$

Proof. For the function (15) with $\{m_j = m_j^*\}_{j=1}^4$ defined by (6), compute the characteristic polynomial of the Hessian matrix $\mathcal{H}(P_*)$:

$$\lambda^3 - L(L+1)V\lambda^2 + L^2\left[(2L-1)V^2 + (L-2)^2 t_2(x_*,y_*,z_*)\right]\lambda - L^3\widetilde{\Phi}_L(x_*,y_*,z_*) .$$

Both expressions t_2 and t_3 are non-negative if the point P_* lies inside the simplex. For $L \in [-1;0]$, the coefficient of λ^2 is not negative, therefore, at least one of the eigenvalues of the Hessian should be non-positive. From this follows the first statement of the theorem. On the contrary, for $L \geq 2$, the three variations in sign in the sequence of the coefficients of the characteristic polynomial certify, due to Descartes rule of signs, that all the eigenvalues of $\mathcal{H}(P_*)$ are positive. Wherefrom follows the last statement of the theorem. □



(a) (b)

Fig. 2. Example 4. Stability domain in the coordinate space (in red).

Example 4. For the simplex (8) and the function $\sum_{j=1}^{4} m_j / |PP_j|^2$, the surface (16) is given by the equation

$$81(x^2+z^2+y^2)^4 - 144\,(-2x^2+2\sqrt{2}xy-y^2+3z^2))(x+\sqrt{2}y)(x^2+z^2+y^2)^2$$

$$+\cdots-304(-2x^2+2\sqrt{2}xy-y^2+3z^2)(x+\sqrt{2}y)-132(x^2+z^2+y^2)+1=0\,.$$

Stability domain $\mathbb{S}$ is bounded by a closed tetrahedron-looking part of this surface surrounding the origin (Fig. 2).

5 Conclusion

We have treated the problem of structure specification for the set of stationary points of the function (1). We have also keened in establishing the influence of the involved parameters onto this set. The suggested solution results in the pair of sets. The one in parameter space absorbs all the bifurcation values while the other in the space of variables contains all the possible locations for the points of minimum. Both sets has been represented analytically by *algebraic* equations or inequalities. This opportunity is certainly granted only for the case of functions with rational exponents L. Even for this case, the computational complexity of the algorithm grows drastically with the increase of the dimension from $n = 2$ to $n = 3$. The hope to overcome this obstacle is connected with a counterpart in $\mathbb{R}^3$ of Theorem 4.

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