The Origin of Mathematical Induction

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# THE ORIGIN OF MATHEMATICAL INDUCTION.*

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## INTRODUCTION.

A criticism often made of mathematics as a subject of study in our high schools and colleges is that it involves nothing of observation, experiment, and induction as these terms are understood in the natural sciences. Whether or not the old and well-developed branches of mathematics as taught in our schools have been made into such well-organized deductive disciplines that this criticism is just, it is true that the work of the original investigators who have developed mathematical science has involved a great deal of observation, experiment, and induction; induction being, according to the Century Dictionary, "the process of drawing a general conclusion from particular cases." Observation and experiment in mathematics do not involve costly and complicated apparatus as often in physics, astronomy, and the other sciences, pencil and paper being all that is ordinarily necessary, but they are just as truly observation and experiment.

In the natural sciences a law arrived at by observation and experiment has to be verified by subsequent experiment by the same or other observers, either directly by repetition of the same experiment or indirectly by testing some logical consequence of the law in question. But in mathematics it is often possible to give rigorous demonstrations of theorems arrived at by ordinary induction. One method of clinching an argument by ordinary induction is what has been called *mathematical induction.* A more significant name and one that is being used more and more is *complete induction.* It is not a method of discovery but a method of proving rigorously that which has already been discovered. It is one of the most fruitful methods in all mathematics. It has

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applications in widely differing branches of mathematics, in algebra, trigonometry, calculus, theory of probability, theory of groups, etc. In American college algebras it is used in proving divisibility theorems like *$x^n - y^n$ is divisible by $x - y$ for all positive integral values of $n$*; in proving the binomial theorem for positive integral exponents; and in proving given formulas for the summation of series, like

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \tfrac{1}{4}n^2(n + 1)^2.$$

A theorem provable by complete induction involves a statement about an integer, usually denoted by $n$, which is to be proved true for all values of the integer. The proof is in two parts. The first part proves that the theorem is true in a special case, that is for a special value of the integer $n$ involved in the theorem. The second part of the proof is what has been called the argument from $n$ to $n + 1$. It is the argument which justifies one in drawing a general conclusion from the special cases verified. For this reason it may properly be called the *induction argument*. The method is too well known to need any further explanation here. If one assumes that the reader understands the method it is not necessary every time to write out explicitly the argument by which the two parts of the proof taken together establish the theorem in question for all values of $n$. It is necessary only to exhibit both parts of the proof. Indeed it is quite customary for writers of mathematical books to give only the argument from $n$ to $n + 1$ and to leave the rest to be supplied by the reader.

## Maurolycus's Use of Complete Induction.

Cantor in his *Vorlesungen über Geschichte der Mathematik*[1] says that Pascal was the originator of the method of complete induction.[2] But he has corrected this statement in a brief note in the *Zeitschrift für Mathematischen und Naturwissenschaftlichen Unterricht*.[3] In this note he says that he has been informed by G. Vacca that Maurolycus[4] described and used the method in his arithmetic which was published in 1575. I quote from Cantor: "Ich wurde durch Herrn G. Vacca darauf aufmerksam gemacht dass schon Maurolycus in seiner Arithmetik von 1575 die Methode genau geschildert und von ihr Gebrauch gemacht hat. Aus Maurolycus aber entnahm sie erst Pascal. Darüber kann nicht der leiseste Zweifel obwalten da Pascal sich 1659 für den Satz

$$2\left[\frac{n(n + 1)}{2}\right] - n = n^2$$

ausdrücklich auf Maurolycus beruft welcher gerade diesen Satz mittels vollständiger Induktion bewiesen hat."[5]

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[1] Vol. II, p. 749.

[2] W. W. R. Ball's *History of Mathematics* has nothing to say about the origin of complete induction.

[3] Vol. XXXIII (1902), p. 536.

[4] *D. Francisci Maurolyci, Abbatis Messanensis, Mathematici Celeberrimi, Arithmeticorum Libri Duo.* Venice, 1575.

[5] The theorem referred to is equivalent to *twice the nth triangular number minus n equals $n^2$*. See page 202 of this paper.

Maurolycus in Book I of his arithmetic begins with the definitions of different kinds of numbers, namely, *even, odd, triangular, square, numeri parte altera longiores,* etc. By definition the $n$th triangular number is the sum of the integers from 1 to $n$ inclusive and the $n$th *numerus parte altera longior* is $n(n - 1)$. He arranges them in a table as follows:

| *Integers* | *Even* | *Odd* | *Triangular* | *Square* | *N. P. A. L.*[1] |
|---|---|---|---|---|---|
| 1 | 0 | 1 | 1 | 1 | 0 |
| 2 | 2 | 3 | 3 | 4 | 2 |
| 3 | 4 | 5 | 6 | 9 | 6 |
| 4 | 6 | 7 | 10 | 16 | 12 |
| 5 | 8 | 9 | 15 | 25 | 20 |
| 6 | 10 | 11 | 21 | 36 | 30 |
| 7 | 12 | 13 | 28 | 49 | 42 |
| . | . | . | . | . | . |
| . | . | . | . | . | . |
| . | . | . | . | . | . |
| $n$ | $E$ | $O$ | $T$ | $S$ | $L$ |

I have added at the bottom of each column a symbol for the $n$th number of that column which I find convenient in explaining the work of Maurolycus. Numbers in the same row Maurolycus calls collateral numbers. A number in one row is said by him to *precede* any number in the following row and to *follow* any number in the preceding row; *e. g.*, he calls 15 the triangular number following the even number 6.

I quote a number of Maurolycus's theorems for reference. In some cases I give the proofs as given by him. The numbering of the theorems is his.

*Proposition IV.* "The odd numbers are obtained from unity by successive additions of 2." (Maurolycus uses this in *Proposition VI* in the form $O_n + 2 = O_{n+1}$, *i. e.*, the $n$th odd number plus 2 equals the next odd number.)

*Proposition VI.* "Every integer plus the preceding integer equals the collateral odd number." [In symbols this is $n + (n - 1) = O_n$.] Maurolycus's proof, freely translated, is this:

> "The integer 2 added to unity makes the integer 3 but when added to 3 it makes an amount greater by 2 and this (by virtue of *Proposition IV*) is the next odd integer, namely 5. Again since the integer 3 added to 2 makes 5, which is the collateral odd integer, when it is added to 4 the result will be greater by 2, that is (by virtue of *Proposition IV*), it will be the next odd integer which is 7. And in like manner to infinity as the proposition states."

This is not a very clear statement of a proof by mathematical induction but the idea is there. Maurolycus's ideas might be put more clearly as follows: The theorem is true by inspection in the case of the first two integers 1 and 2, *i. e.*, $2 + 1 = 3$ which is the odd integer collateral to 2. This is the first part of the induction proof. Maurolycus then takes up the special cases $3 + 2 = 5$ and $4 + 3 = 7$ and in doing so he shows by his repeated use of *Proposition IV* that

[1] *Numeri Parte Altera Longiores.*

the other part of the mathematical induction proof was in his mind. His *Proposition IV* furnishes the argument from $n$ to $n+1$. In modern notation it would be put in this way:

If $n + (n - 1) = O_n$ (*i. e.*, if any integer plus the preceding one equals the collateral odd integer), the result of adding $1 + 1$ to the left side and 2 to the right side is $(n + 1) + n = O_n + 2$. But by Prop. IV, $O_n + 2 = O_{n+1}$. Therefore $(n + 1) + n = O_{n+1}$.

This argument from $n$ to $n+1$ seems to have been in Maurolycus's mind. But if this were the only example of complete induction in his work it might not be a conclusive proof that he understood the method. *Proposition XV* is a much more convincing case. But before giving an account of it I wish to state several other of his theorems for reference and to discuss the proposition which Cantor says Pascal got from Maurolycus.

*Proposition VIII.* "Every triangular number doubled equals the following *numerus parte altera longior*." (In symbols this is $2T_n = L_{n+1}$.)

*Proposition X.* "Every *numerus parte altera longior* plus its collateral integer equals the collateral square number." (In symbols, $L_n + n = S_n$.)

*Proposition XI.* "Every triangular number plus the preceding triangular number equals the collateral square number." (In symbols, $T_n + T_{n-1} = S_n$.)

This proposition, although stated somewhat differently by Cantor, is the one which Cantor says Pascal got from Maurolycus and which he says Maurolycus proved by complete induction. For since a triangular number is equal to the sum of the natural numbers in order, $T_{n-1} = T_n - n$, and it follows that $T_n + T_{n-1} = 2T_n - n$; or, since by the formula for the sum of an arithmetic progression[1] the $n$th triangular number is $[n(n+1)]/2$, the equation $T_n + T_{n-1} = S_n$ is equivalent to

$$2\left[\frac{n(n+1)}{2}\right] - n = S_n = n^2,$$

which is the form that Cantor gives. But Cantor is wrong in saying that this theorem was proved by Maurolycus by complete induction. For Maurolycus's proof (in modern notation) is this:

By definition $T_n = T_{n-1} + n$. Therefore $T_n + T_{n-1}$, the left-hand member of the relation to be proved, is equal to $2T_{n-1} + n$ which equals $L_n + n$ (by *Proposition VIII*), and this equals $S_n$ (by *Proposition X*).

*Proposition XIII.* "Every square number plus the following odd number equals the following square number." (In symbols, $S_n + O_{n+1} = S_{n+1}$.)

*Proposition XV.* "The sum of the first $n$ odd integers is equal to the $n$th square number."[2] (In symbols, $O_1 + O_2 + O_3 + \cdots + O_n = S_n$.) Maurolycus's proof freely translated is this:

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[1] In *Proposition VII*, which I have not quoted, Maurolycus uses the usual arithmetical progression device for proving $T_n = [n(n+1)]/2$. He says in effect: $T_n = 1 + 2 + 3 + \cdots + n$ and also $T_n = n + (n - 1) + (n - 2) + \cdots + 2 + 1$, and therefore by addition $2T_n = n(n + 1)$.

[2] This theorem in the form $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ is often given in American college algebras as one of the first examples of complete induction to be proved by the student.

"By a previous proposition[1] the first square number (unity) added to the following odd number (3) makes the following square number (4); and this second square number (4) added to the 3d odd number (5) makes the 3d square number (9); and likewise the 3d square number (9) added to the 4th odd number (7) makes the 4th square number (16); and so successively to infinity the proposition is demonstrated by the repeated application of *Proposition XIII*."

This is a clear case of a complete induction proof. *Proposition XIII* is used as a lemma. It furnishes the argument from $n$ to $n + 1$. The first few special cases are mentioned in *Proposition XV* itself. In modern symbols the proof would be this:

*1st.* The theorem is true when $n = 1$. *2d.* Assume that it is true when $n = k$, *i. e.*, assume $O_1 + O_2 + \cdots + O_k = S_k$; add $O_{k+1}$ to both sides of this equation and get $O_1 + O_2 + \cdots + O_{k+1} = S_k + O_{k+1}$ which equals $S_{k+1}$, by *Proposition XIII*.

## Pascal's Use of Complete Induction.

Pascal repeatedly used the method of complete induction in connection with his arithmetical triangle[2] and its applications. Cantor's argument that Pascal borrowed the method from Maurolycus is valid in spite of the fact that he is in error in saying that Maurolycus proved *Proposition XI* of his arithmetic by complete induction. For Pascal does refer to Maurolycus for a proof of this proposition which shows that Pascal was familiar with the part of Maurolycus's arithmetic in which Maurolycus does use complete induction. It is in a letter from Pascal (writing under the pseudonym Amos Dettonville) to Carcavi[3] that Pascal refers to Maurolycus for the proof of the theorem that "*twice the nth triangular number minus n equals $n^2$.*" Pascal says "*Cela est aisé par Maurolic.*" Pascal makes use of this theorem in connection with his work on centers of gravity.

I give the following as two interesting examples of Pascal's use of the method of complete induction. I use modern algebraic notation for the sake of brevity. Without modern notation it would be necessary to explain the construction of Pascal's arithmetical triangle and many of his theorems about it.

Theorem.[4] *The number of combinations of n things k at a time is to the number of combinations of n things k + 1 at a time as k + 1 is to n − k.* (In symbols, ${}_nC_k : {}_nC_{k+1} = k + 1 : (n - k.)$)

*Proof.—First part.* By inspection the theorem is true when $n = 2$. For then the only possible values of $k$ and $k + 1$ are 1 and 2 respectively and ${}_2C_1 : {}_2C_2 = 2 : 1$.

*Second part.* Assume that the theorem is true when $n = q$. That is, assume[5]

$$(A) \qquad {}_qC_k : {}_qC_{k+1} = k + 1 : q - k$$

for all positive integral values of $k < q$. It is then to be proved that, on this assumption,

$$(B) \qquad {}_{q+1}C_j : {}_{q+1}C_{j+1} = j + 1 : q + 1 - j$$

for all positive integral values of $j < q + 1$.

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[1] He refers to *Proposition XIII*.

[2] *Oeuvres Complètes de Blaise Pascal*, Paris, 1889, Vol. III, p. 243 ff.

[3] L. c., p. 376.

[4] This is Pascal's "Consequence XII," l. c., p. 248.

[5] This relation $(A)$ is seen by inspection to be true when $k = q$, if we define ${}_sC_t = 0$ when $t > s$.

[(*B*) is obtained from (*A*) by replacing $q$ in (*A*) by $q + 1$ and by using another letter for $k$ to avoid confusion later.] The well-known relation[1]

$$(C) \qquad {}_{N}C_{R} = {}_{N-1}C_{R-1} + {}_{N-1}C_{R}$$

is needed to prove that (*B*) follows from (*A*).

By relation (*C*) the left-hand member of (*B*) is equal to

$$\frac{{}_{q}C_{j-1} + {}_{q}C_{j}}{{}_{q}C_{j} + {}_{q}C_{j+1}} = \frac{\dfrac{{}_{q}C_{j-1}}{{}_{q}C_{j}} + 1}{1 + \dfrac{{}_{q}C_{j+1}}{{}_{q}C_{j}}}.$$

On applying relation (*A*) to the minor fractions ${}_{q}C_{j-1}/{}_{q}C_{j}$ and ${}_{q}C_{j+1}/{}_{q}C_{j}$ this becomes

$$\frac{\dfrac{j}{q-j+1} + 1}{1 + \dfrac{q-j}{j+1}} = \frac{j+1}{q-j+1}$$

which was to be proved.

Another example from Pascal is this problem in the division of stakes in a gambling game. Two players *A* and *B* of equal skill, playing for a stake *P*, wish to leave the game table before finishing their game. Their scores and the number of points which constitute the game being given indirectly as follows: Player *A* lacks $\alpha$ points of winning and player *B* lacks $\beta$ points. If $\alpha + \beta$ be denoted by $n$, Pascal says that the stakes should be divided between *B* and *A* in the ratio

$$({}_{n-1}C_0 + {}_{n-1}C_1 + {}_{n-1}C_2 + \cdots + {}_{n-1}C_{\alpha-1}) : ({}_{n-1}C_\alpha + {}_{n-1}C_{\alpha+1} + \cdots + {}_{n-1}C_{n-1}).$$

Since[2]

$${}_{n-1}C_0 + {}_{n-1}C_1 + {}_{n-1}C_2 + \cdots + {}_{n-1}C_{n-1} = 2^{n-1},$$

this is the same as saying that *B*'s share of the whole stake is

$$\frac{P}{2^{n-1}}[{}_{n-1}C_0 + {}_{n-1}C_1 + {}_{n-1}C_2 + \cdots + {}_{n-1}C_{n-1}],$$

and *A*'s share is

$$\frac{P}{2^{n-1}}[{}_{n-1}C_\alpha + {}_{n-1}C_{\alpha+1} + \cdots + {}_{n-1}C_{n-1}].$$

*Proof.—First Part.* The theorem is true in the special case in which $n = 2$. For in this case the scores of *A* and *B* are even and each lacks only one point of winning. They are of equal skill and so one is as likely to win the game as the other and the stake should be divided in the ratio 1 to 1. The theorem states that the stake should be divided in the ratio ${}_1C_0 : {}_1C_1$, which is $1 : 1$. The theorem is also true in the special case in which $n = 3$. In this case the score at the end of play must be such that one player lacks one point and the other lacks two points. Suppose that it is *A* who lacks the one point. Then *B* lacks two points. If the play were to continue for one more point and if *A* were to win that point he would win the game and be entitled to the whole stake *P*. But if he were to lose he would be entitled to $P/2$ by virtue of the special

[1] Fine, H. B., *College Algebra*, p. 404. This relation is true when $R = 0$ and when $N \geqq R$ if we define ${}_sC_0 = 1$ and ${}_sC_t = 0$ when $t > s$.

[2] It is customary to give ${}_NC_0$ the meaning ${}_NC_0 = 1$ by definition. The relation ${}_{n-1}C_0 + {}_{n-1}C_1 + \cdots + {}_{n-1}C_{n-1} = 2^{n-1}$ is more usually written ${}_{n-1}C_1 + {}_{n-1}C_2 + \cdots + {}_{n-1}C_{n-1} = 2^{n-1} - 1$. See Fine, H. B., *College Algebra*, p. 402.

case previously considered. The division of the stake should therefore be such that $A$ will get the $P/2$ which he is entitled to in case he loses the next point and one half of the other $P/2$ which he has an even chance of winning. This is the same as saying that $A$ should receive the arithmetic mean between $P$ and $P/2$. The stake should therefore be divided between $B$ and $A$ in the ratio 1 to 3. The theorem gives the ratio ${}_2C_0 : ({}_2C_1 + {}_2C_2)$, which is $1 : 3$.

If $A$ lacked 2 points and $B$ one point the division between $B$ and $A$ would be in the ratio $3 : 1$. The theorem in this case gives $({}_2C_0 + {}_2C_1) : {}_2C_2$ which is $3 : 1$.

*Second Part of the Proof.* Assume the theorem true for $n$, *i. e.*, assume that when the points lacking for $A$ and $B$ have their sum equal to $n$ the equitable division of the stake is as the theorem indicates. To prove that on this assumption the theorem is true when the sum of the points lacking for $A$ and $B$ is $n + 1$, suppose that $A$ lacks $k$ points and $B$ lacks $l$ points where $k + l = n + 1$. If the play were to continue and $A$ were to win the next point, the sum of the points lacking after that would be exactly $n$ points (*i. e.*, $A$ would lack $k - 1$ points and $B$ would lack $l$ points) and by our assumption the rule given by the theorem may be applied. The result of applying the theorem in this case is

$$\frac{P}{2^{n-1}}[{}_{n-1}C_0 + {}_{n-1}C_1 + \cdots + {}_{n-1}C_{k-2}]$$

for $B$'s share of the stake. (This comes from putting $k - 1$ for $\alpha$ in the statement of the theorem.) But if $A$ were to lose then $B$ would win and $A$ would lack $k$ points and $B$ only $l - 1$ points. The sum of the points lacking would be exactly $n$ and the rule given by the theorem may be applied as before, putting $k$ for $\alpha$. The result is

$$\frac{P}{2^{n-1}}[{}_{n-1}C_0 + {}_{n-1}C_1 + \cdots + {}_{n-1}C_{k-1}]$$

for $B$'s share. As in the special case previously considered $B$'s share of the stake should be the arithmetic mean between

$$\frac{P}{2^{n-1}}\ {}_{n-1}C_0 + {}_{n-1}C_1 + \cdots + {}_{n-1}C_{k-2}] \quad \text{and} \quad \frac{P}{2^{n-1}}[{}_{n-1}C_0 + {}_{n-1}C_1 + \cdots + {}_{n-1}C_{k-1}]$$

which equals

$$\frac{P}{2^n}[2({}_{n-1}C_0 + {}_{n-1}C_1 + \cdots + {}_{n-1}C_{k-2}) + {}_{n-1}C_{k-1}].$$

This may be written in the form

$$\frac{P}{2^n}[{}_{n-1}C_0 + ({}_{n-1}C_0 + {}_{n-1}C_1) + ({}_{n-1}C_1 + {}_{n-1}C_2) + \cdots + ({}_{n-1}C_{k-3} + {}_{n-1}C_{k-2}) + ({}_{n-1}C_{k-2} + {}_{n-1}C_{k-1})].$$

But by virtue of the relation ${}_NC_R = {}_{N-1}C_{R-1} + {}_{N-1}C_R$, used in the proof of the preceding theorem of Pascal's, each binomial in this expression may be replaced by a single term so that $B$'s share is

$$\frac{P}{2^n}[{}_nC_0 + {}_nC_1 + {}_nC_2 + \cdots + {}_nC_{k-2} + {}_nC_{k-1}]$$

which is just what the theorem gives for $B$'s share.

This completes the induction argument for this theorem of Pascal's.

## Other and More Recent Uses of Complete Induction.

The following examples will give some idea of the variety of uses which can be made of the method of complete induction.

1. (*a*) $x^n - y^n$ *is divisible by* $x - y$.

(*b*) $x^n - y^n$ *is divisible by* $x + y$ *when* $n$ *is even but not*[1] *when* $n$ *is odd.*

(*c*) $x^n + y^n$ *is divisible by* $x + y$ *when* $n$ *is odd but not*[1] *when* $n$ *is even.*

[1] The reference here (as elsewhere in the paper) is to *algebraic* divisibility. The statements obviously need not be true for the case of divisibility of the integers represented by the forms.

Negative divisibility theorems like the second parts of ($b$) and ($c$) are just as easily proved by complete induction as positive divisibility theorems such as ($a$).

2. $$1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \cdots + n(n+1) = \tfrac{1}{3}n(n+1)(n+2).$$

American college algebras contain many such examples in the summation of series.

3. *De Moivre's Theorem.* $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ *for positive integral values of* $n$.

4. *The binomial theorem for positive integral exponents.*

5. *If*

$$F_n(x) \equiv (x+a_1)(x+a_2)(x+a_3)\cdots(x+a_n)$$
$$\equiv x^n + A_1x^{n-1} + A_2x^{n-2} + \cdots + A_{n-1}x + A_n,$$

*then*

$$A_1 = \Sigma a_1, \quad A_2 = \Sigma a_1a_2, \quad A_3 = \Sigma a_1a_2a_3, \quad \cdots, \quad A_n = a_1a_2\cdots a_n.$$

(See H. Weber and J. Wellstein, *Encyclopädie der Elementar-Mathematik,* volume I, p. 196.)

6. *Fermat's Theorem.* $a^n - a$ *is divisible by* $n$ *if* $a$ *is any integer and* $n$ *is a prime integer.* (See Weber and Wellstein, volume I, p. 197.)

7. *The Polynomial Theorem.*

$$(x+y+z+\cdots)^n = \Sigma\frac{n!}{\alpha!\beta!\gamma!\cdots}\,x^\alpha y^\beta z^\gamma\cdots.$$

(See Weber and Wellstein, volume I, p. 198.)

8. *Any polynomial symmetric in* $x_1, x_2, \cdots, x_n$ *is equal to a polynomial in the elementary symmetric functions.* (See Weber and Wellstein, volume I, p. 232.)

9. *A necessary and sufficient condition that a polynomial in any number of variables vanish identically is that all its coefficients are zero.* (For a proof of this theorem for one variable and its extension by complete induction see Maxime Bôcher's *Introduction to Higher Algebra,* p. 5.)

10. *If* $f_1$ *and* $f_2$ *are polynomials in any number of variables of degrees* $m_1$ *and* $m_2$ *respectively, the product* $f_1f_2$ *will be of degree* $m_1m_2$. (See Bôcher's *Higher Algebra,* p. 6.)

11. *A necessary and sufficient condition that a polynomial* $f(x_1x_2\cdots x_n)$ *vanish identically is that it vanish throughout the neighborhood of a point* $(a_1a_2\cdots a_n)$. (See Bôcher's *Higher Algebra,* p. 10.)

12. *If* $f(x)$ *is a polynomial of the nth degree,*

$$f(x) \equiv a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n \qquad (a_0 \neq 0),$$

*there exists one and only one set of constants* $\alpha_1\alpha_2\cdots\alpha_n$ *such that*

$$f(x) \equiv a_0(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n).$$

To prove this theorem by complete induction one needs the fundamental theorem of algebra that there is at least one value of $x$ for which such a polynomial as $f(x)$ vanishes. (See Bôcher's *Higher Algebra,* p. 17.)

13. *If all the* $(r+1)$*-rowed principal minors of the symmetrical matrix*

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \qquad (\text{where } a_{ij} = a_{ji})$$

*are zero, and also all the* $(r+2)$*-rowed principal minors, then the rank of the matrix is* $r$ *or less.* (See Bôcher's *Higher Algebra,* p. 57.)

14. *The conjugate of the product of any number of matrices is the product of their conjugates taken in reverse order.* (See Bôcher's *Higher Algebra,* p. 65.)

15. *If* $y = u_1u_2\cdots u_n$ *and* $y'$, $u_1'$, $u_2'$, *etc., denote first derivatives with respect to a variable* $x$, *then*

$$\frac{y'}{y} = \frac{u_1'}{u_1} + \frac{u_2'}{u_2} + \cdots + \frac{u_n'}{u_n}.$$

16. *Leibnitz's Theorem.*

$$\frac{d^n(uv)}{dx^n} = u_nv + nu_{n-1}v_1 + \frac{n(n-1)}{1\cdot 2}u_{n-2}v^2 + \cdots + nu_1v_{n-1} + uv_n,$$

*where* $u$ *and* $v$ *are functions of* $x$ *and the subscripts denote derivatives with respect to* $x$. *That is,*

$$u_1 = \frac{du}{dx}, \qquad u_2 = \frac{d^2u}{dx^2}, \qquad \text{etc.}$$

(For a proof of this theorem and some examples of its use see G. A. Osborne's *Differential and Integral Calculus,* pages 65–67.)

17. (*a*) *Limit* $(x_1 + x_2 \cdots + x_n) =$ *limit* $x_1 +$ *limit* $x_2 + \cdots +$ *limit* $x_n$.

(*b*) *Limit* $(x_1x_2\cdots x_n) =$ (*limit* $x_1$)(*limit* $x_2$)$\cdots$(*limit* $x_n$).

(See W. F. Osgood's *Differential and Integral Calculus,* p. 16.)

18. *If*

$$y = \log x, \qquad \frac{d^ny}{dx^n} = \frac{(-1)^{n-1}(n-1)!}{x^n}.$$

Numerous examples like this are to be found in G. A. Osborne's *Differential Calculus,* pages 62–65.